Lecture 6: Overlapping Generations Model

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Roadmap

Motivation

- In our models so far, we've had infinitely-lived representative households.

- What happens if we relax this assumption? Why?

Preview of the Model

- We'll develop a model with "young" and "old" generations in each period.
- Each generation will live for two time periods and die.
- Start from $t = 0$ (with initial old and young) and time goes on forever.
- Receive an endowment of perishable consumption goods in each period.
- Endowment is larger when young than when old.
- Motive for consumption smoothing, in principle, gives motivation for trading between young and old.

Preview of the Punchline

- In the absence of some tradable asset, the competitive equilibrium of this model will not be Pareto optimal.
- Age structure of the model creates problems.
- Why would the young ever give any of their endowment to the old?
- Money can move us towards the efficient allocation.

Roadmap

Setup

- Young and old each period.
- N_t new people are born each period for $t = 0, 1, \ldots$.
- Period $t = 0$ starts with an initial old of mass N_{-1}.
- At period $t = 0$, total number of people is $N_{-1} + N_0$.
- For general t, total people is $N_{t-1} + N_t$.

Setup

- Young are given endowment of 2 goods each period; old 1 good.
- Total endowment of consumption goods at t given by $N_{t-1} + 2N_t$.
- Assume time separable and concave utility function.
- Concavity generates motivation for consumption in both periods of life.
- Two types of households: those born from $t = 0$ onwards and the initial old.
- Households can hold one period bonds b_t that offer gross return $R_t > 1$.
- Assume for simplicity that $N_t = N \ \forall t$.

Utility Function

- Assume log utility

$$U(c_{1,t}, c_{2,t}) = \log(c_{1,t}) + \beta \log(c_{2,t})$$

where $0 < \beta < 1$ is the discount factor and $c_{j,t}$ for $j \in \{1,2\}$ denotes consumption in the j^{th} period of life for a household born in period t.

Household Problem: $t = 0$ Onwards

- A household born in $t = 0$ or later solves the problem

$$\max_{\{c_{1,t}, c_{2,t}, b_t\}} \log(c_{1,t}) + \beta \log(c_{2,t})$$

subject to budget constraints in each period

$$c_{1,t} + b_t \leq 2$$
$$c_{2,t} \leq R_t b_t + 1$$

Household Problem: Initial Old

- The initial old solve the problem

$$\max_{\{c_{2,-1}\}} \log(c_{2,-1})$$

subject to budget constraints in each period

$$c_{2,t} \leq 1$$

Roadmap

Household Solution: $t = 0$ Onwards

- Substitute the budget constraints into the utility function to get the objective

$$\mathcal{L} = \log(2 - b_t) + \beta \log(R_t b_t + 1)$$

  why do the two constraints bind?

- FOC

$$\frac{\partial \mathcal{L}}{\partial b_t} = 0 \Rightarrow -\frac{1}{2 - b_t} + \frac{\beta R_t}{R_t b_t + 1} = 0$$
$$\Rightarrow b_t = \frac{2R_t\beta - 1}{R_t[1 + 2\beta]}$$

Household Solution: Initial Old

- Simply consume your endowment

$$c_{2,-1} = 1$$

Equilibrium Definition

- A competitive equilibrium of the OLG model is defined such that
  - All agents optimise subject to budget constraints.
  - Prices all taken as given.
  - Bond market clears: the sum of bond holdings across all households equals zero, (known as zero net supply).
    - Says that the savers and borrowers cancel each other out.
  - Goods market clears: sum of total consumption equals sum of total endowments.

Bond Market Clearing

- See that since all the households born in a given period are identical, it follows that $b_t = 0 \ \forall t$.
- No lending/borrowing takes place.
- The equilibrium R_t consistent with this is such that

$$b_t = 0$$
$$\Rightarrow R_t = \frac{1}{2\beta}.$$

What does this mean? It's the discounted marginal utility ratio at the endowment points.

Autarky

- Equilibrium is autarky.
- We simply consume our endowments.
- No trade between generations as the old can't pay back the young next period!

Pareto Improvements

- A Pareto improvement takes place when, through reallocation, we make one generation better-off without making another worse-off.
- In autarky, the initial old and future generations receive utilities of $\log(1) = 0$ and $\log(2) + \beta \log(1) = \log(2)$ respectively.
- A Pareto improvement would then take place if we could re-allocate resources such that initial old get utility above zero and future generations get at least $\log(2)$ without violating resource constraint.

Pareto Improvements

- Resource constraint here is $c_{1,t} + c_{2,t-1} = 3$. Why?
- Consider re-distributing all 2 of the young's endowment each period to the old.
- Initial old get utility $\log(3)$.
- All future generations get lifetime utility $\beta \log(3)$.
- This represents a Pareto improvement when

$$\beta \log(3) > \log(2)$$
$$\Rightarrow \beta > \frac{\log(2)}{\log(3)} \approx 0.64.$$

- Improvement if all future generations are "patient enough".
- This competitive equilibrium is inefficient.

Pareto Improvements

- How can we facilitate such a re-distribution from young to old in each period?
- There are several approaches, which involve government intervention.
- We'll focus here on using fiat money.

Roadmap

Including Money

- Say that the central bank issues $\mathcal{M}$ units of money at $t = 0$ amongst the initial old.
- Each initial old receives $\frac{\mathcal{M}}{N}$ of cash.
- Assume that the money supply then stays constant forever.

Household Problem: $t = 0$ Onwards

- A household born in $t = 0$ or later solves the problem

$$\max_{\{c_{1,t}, c_{2,t}, m_t\}} \log(c_{1,t}) + \beta \log(c_{2,t})$$

where m_t is the fraction of their endowment they sell. They solve the problem subject to budget constraints in each period

$$c_{1,t} + \phi_t m_t \leq 2$$
$$c_{2,t} \leq \phi_{t+1} m_t + 1$$

where ϕ_t is the amount of consumption goods that a unit of money can buy in t (the inverse of the goods price).

Household Problem: Initial Old

- The initial old solve the problem

$$\max_{\{c_{2,-1}\}} \log(c_{2,-1})$$

subject to budget constraints

$$c_{2,-1} \leq 1 + \phi_0 \frac{\mathcal{M}}{N}$$

Roadmap

Household Solution: $t = 0$ Onwards

- Substitute the budget constraints into the utility function to get the objective

$$\mathcal{L} = \log(2 - \phi_t m_t) + \beta \log(\phi_{t+1} m_t + 1)$$

- FOC

$$\frac{\partial \mathcal{L}}{\partial m_t} = 0 \Rightarrow -\frac{\phi_t}{2 - \phi_t m_t} + \frac{\beta \phi_{t+1}}{\phi_{t+1} m_t + 1} = 0 \tag{1}$$

$$\Rightarrow m_t = \frac{2\phi_{t+1}\beta - \phi_t}{[1 + \beta]\phi_t \phi_{t+1}}$$

Household Solution: Initial Old

- Simply consume your endowment

$$c_{2,-1} = 1 + \phi_0 \frac{\mathcal{M}}{N}$$

Equilibrium Definition

- A competitive equilibrium of the OLG model with money is defined such that
  - All agents optimise subject to budget constraints.
  - Prices all taken as given.
  - Money market clears: aggregate money demand equals $\mathcal{M}$.
  - Goods market clears: sum of total consumption equals sum of total endowments.

Money Market Clearing

- The net supply of money is $\mathcal{M}$. How does this differ from bonds in the earlier version of the model? What does this mean?
- Market clearing requires that

$$\mathcal{M} = m_t N \tag{2}$$
$$\Rightarrow m_t = \frac{\mathcal{M}}{N}$$

- Using (1)

$$\frac{2\phi_{t+1}\beta - \phi_t}{[1+\beta]\phi_t\phi_{t+1}} = \frac{\mathcal{M}}{N} \tag{3}$$
$$\Rightarrow \phi_t = \frac{2\phi_{t+1}\beta\frac{N}{\mathcal{M}}}{(1+\beta)\phi_{t+1} + \frac{N}{\mathcal{M}}}$$

which is increasing in ϕ_{t+1}, decreasing in the fixed supply of money and increasing in the population size.

Value of Money

- Let's look for an equilibrium, which is stationary.
- This means that for all future generations, $c_{1,t} = \bar{c}_1$ and $c_{2,t} = \bar{c}_2$.
- By (2), it must be that m_t is also constant.
- Follows from the period budget constraints that ϕ_t is also constant at some price $\bar{\phi}$.

Value of Money

- Can then solve equation (3) for $\bar{\phi}$

$$\bar{\phi} = \left(\frac{N}{\mathcal{M}}\right) \frac{[2\beta - 1]}{1 + \beta}$$

which is positive if $\beta > 1/2$: if agents are sufficiently "patient".

- Given this, it follows that the value of money holdings are given by

$$\bar{\phi}\bar{m} = \frac{2\beta - 1}{1 + \beta}$$

exercise: check.

Money and Welfare

- The values of consumption for future generations are

$$\bar{c}_1 = \frac{3}{1+\beta}$$
$$\bar{c}_2 = \frac{3\beta}{1+\beta}$$

which yields welfare of

$$\log\left(\frac{3}{1+\beta}\right) + \beta \log\left(\frac{3\beta}{1+\beta}\right)$$

which is always weakly above the autarky welfare (exercise: show).

Money and Welfare

- The consumption of initial old is then given by

$$
\begin{aligned}
c_{2,-1} &= 1 + \left(\frac{N}{\mathcal{M}}\right) \frac{[2\beta - 1]}{1+\beta} \frac{\mathcal{M}}{N} \\
&= 1 + \frac{2\beta - 1}{1+\beta}
\end{aligned}
$$

  which is greater than unity if money has value.

- The monetary equilibrium yields a Pareto improvement over the autarkic equilibrium!

- Why?

Roadmap

Takeaways

- OLG model is a workhorse of macro.
- Timing structure for generations can lead to inefficiency.
- Money in this model can yield Pareto improvements.
- In fact, the monetary equilibrium we found **is** the Pareto optimal allocation (exercise).