Lecture 4: Money in the Utility Function

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Roadmap

Recap

- Last time we studied an entirely real model.
- No role for nominal variables.
- Now let's think about money.
- Need money demand and supply.
- Fastest and dirtiest way to generate money demand is to stick it in the household's utility function, (like a type of good in the model).
- Sidrauski, M. (1967), "Inflation and Economic Growth", *Journal of Political Economy*, 75, pp. 796 – 810.

Roadmap

Household Setup

- Let's forget about capital for now.
- Assume that households can hold cash m_{t+1} or discount bonds in each period b_{t+1}.
- Otherwise the setup is the same as the RBC model.

Period Utility Function

- Household's period utility function

$$\frac{c_t^{1-\sigma}}{1-\sigma} + \frac{(m_{t+1}/p_t)^{1-\nu}}{1-\nu} - \frac{n_t^{1+\psi}}{1+\psi}$$

where m_{t+1}/p_t represents real balances.

- What does this mean?

Period Utility Function

Floyd "Money" Mayweather Jnr.
Former world boxing champion

Period Utility Function

Plaza Bar, Madison WI (USA)
$2.50 Long Island Iced Teas on Thursdays

Household's Problem

- Problem:

$$\max_{\{c_t, n_t, b_{t+1}, m_{t+1}\}_{t=0}^{\infty}} \mathbb{E}_0 \sum_{t=0}^{\infty} \beta^t \left[\frac{c_t^{1-\sigma}}{1-\sigma} + \frac{(m_{t+1}/p_t)^{1-\nu}}{1-\nu} - \frac{n_t^{1+\psi}}{1+\psi} \right]$$

subject to their budget constraints

$$p_t c_t + q_t b_{t+1} + m_{t+1} \leq w_t n_t + m_t + b_t + d_t$$

$$b_0, m_0 \text{ given}$$

$$\lim_{T \to \infty} [m_{T+1} + b_{T+1}] = 0.$$

Firm's Problem

- Problem:

$$\max_{\{n_t\}} d_t = p_t y_t - w_t n_t$$

where the production function is given by

$$y_t = a_t n_t^{\alpha}$$

- Recall that we abstract from capital.
- The firm takes price and wage as given.

Monetary Authority

- Assume that the central bank sets an exogenous supply of money $\{m_t\}_{t=0}^{\infty}$.
- Money law of motion

$$m_{t+1} = e^{\zeta_{t+1}} m_t \tag{1}$$

where

$$\zeta_{t+1} = \rho_\zeta \zeta_t + \epsilon_{\zeta,t+1}, \;\; \epsilon_{\zeta,t+1} \sim N(0, \sigma_\zeta^2). \tag{2}$$

- Can be re-written in terms of real balances as

$$\frac{m_{t+1}}{p_t} = e^{\zeta_{t+1}} \frac{m_t}{p_{t-1}} \frac{1}{\pi_t}$$

where $\pi_t = \frac{p_t}{p_{t-1}}$ is the gross inflation rate at t.

Roadmap

Household Optimality

- Lagrangian given by

$$\mathcal{L} = \mathbb{E}_0 \sum_{t=0}^{\infty} \beta^t \left[\frac{c_t^{1-\sigma}}{1-\sigma} + \frac{(m_{t+1}/p_t)^{1-\nu}}{1-\nu} - \frac{n_t^{1+\psi}}{1+\psi} \right] + \mathbb{E}_0 \sum_{t=0}^{\infty} \lambda_t \left[w_t n_t + m_t + b_t - m_{t+1} - q_t b_{t+1} - p_t c_t \right]$$

Household Optimality: First Order Conditions

- FOCs:

$$\frac{\partial \mathcal{L}}{\partial c_t} = 0 \Rightarrow \beta^t c_t^{-\sigma} - p_t \lambda_t = 0 \tag{3}$$

$$\frac{\partial \mathcal{L}}{\partial n_t} = 0 \Rightarrow -\beta^t n_t^{\psi} + \lambda_t w_t = 0 \tag{4}$$

$$\frac{\partial \mathcal{L}}{\partial b_{t+1}} = 0 \Rightarrow -q_t \lambda_t + \mathbb{E}_t[\lambda_{t+1}] = 0 \tag{5}$$

$$\frac{\partial \mathcal{L}}{\partial m_{t+1}} = 0 \Rightarrow -\lambda_t + \mathbb{E}_t[\lambda_{t+1}] + \beta^t \frac{1}{p_t} \left(\frac{m_{t+1}}{p_t} \right)^{-\nu} = 0 \tag{6}$$

Firm Optimality: First Order Conditions

- FOC

$$\frac{\partial d_t}{\partial n_t} = 0 \Rightarrow \alpha p_t a_t n_t^{\alpha-1} - w_t = 0 \tag{7}$$

Equilibrium Definition

- The competitive equilibrium of the MIU model is defined as a sequence of prices $\{w_t, p_t, q_t\}_{t=0}^{\infty}$ and allocations $\{c_t, m_{t+1}, b_{t+1}, n_t\}$ with the state vector $\{m_t, a_t, b_t\}$ taken as given by the agents in the model. Optimality conditions (3) – (7) above hold, the no ponzi scheme condition holds and all markets clear.

Canonical Representation

- Money demand from (6) and (3)

$$\mu_{t+1}^{-\nu} c_t^{\sigma} = 1 - q_t$$

where $\mu_{t+1} = m_{t+1}/p_t$ is real balances at t.

- Labour supply (how does this differ from last class?)

$$c_t^{\sigma} n_t^{\psi} = \omega_t$$

where $\omega_t = w_t/p_t$ is the real wage at t.

- Labour demand

$$\alpha a_t n_t^{\alpha-1} = \omega_t$$

Canonical Representation

- Euler equation (how does this differ from last class?)

$$q_t = \mathbb{E}_t \left[\beta \left(\frac{c_{t+1}}{c_t} \right)^{-\sigma} \frac{1}{\pi_{t+1}} \right]$$

where $\pi_{t+1} = p_{t+1}/p_t$ is the gross inflation rate at $t+1$.

- Money supply

$$\frac{\mu_{t+1}}{\mu_t} = e^{\zeta_{t+1}} \frac{1}{\pi_t} \tag{8}$$

- Budget constraint (where is this coming from?)

$$\omega_t n_t + \mu_t \frac{1}{\pi_t} = \mu_{t+1} + c_t$$

- Six equations in six unknowns $\{\mu_{t+1}, c_t, q_t, n_t, \omega_t, \pi_t\}$.

Roadmap

What Affects Money Supply and Demand?

- Money supply

$$\mu_{t+1} = \mu_t e^{\zeta_{t+1}} \frac{1}{\pi_t}$$

- Money demand

$$\mu_{t+1} = \left(\frac{1}{1 - q_t} \right)^{\frac{1}{\nu}} c_t^{\frac{\sigma}{\nu}}$$

- Gross nominal interest rate versus bond price

$$i_t = \frac{1}{q_t}$$

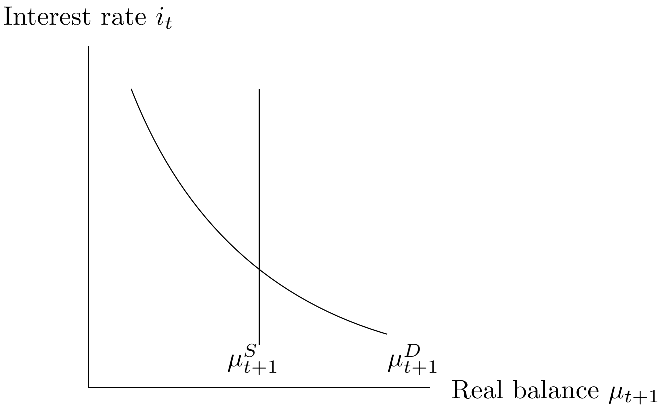
What Affects Money Supply and Demand?

- Money demand in terms of nominal interest rate

$$\mu_{t+1} = \left(\frac{i_t}{i_t - 1} \right)^{\frac{1}{\nu}} c_t^{\frac{\sigma}{\nu}}$$

which is decreasing in i_t for $\nu > 0$.

What Affects Money Supply and Demand?



Roadmap

What Does the Steady State Look Like?

- Steady state money supply equation (8)

$$\frac{\bar{\mu}}{\bar{\mu}} = e^0 \frac{1}{\bar{\pi}}$$
$$\Rightarrow \bar{\pi} = 1$$

given that steady state has shocks shut-down ($\zeta_{t+1} = 0$).

- What is the implication of $\bar{\pi} = 1$?

What Does the Steady State Look Like?

- Steady state Euler equation

$$\bar{q} = \beta \frac{1}{\bar{\pi}}$$

- Steady state labour supply

$$\bar{c}^{\sigma} \bar{n}^{\psi} = \bar{\omega}$$

- Steady state labour demand

$$\alpha \bar{n}^{\alpha - 1} = \bar{\omega}$$

What Does the Steady State Look Like?

- Steady state money demand

$$\bar{\mu} = \left(\frac{\bar{i}}{\bar{i} - 1} \right)^{\frac{1}{\nu}} \bar{c}^{\frac{\sigma}{\nu}}$$

- Steady state budget constraint

$$\bar{\omega} \bar{n} = \bar{c}$$

Steady State Money Neutrality

- None of the real variables depend on money in the long run!
- Referred to as monetary neutrality.

Roadmap

Log-Linearisation

- Log-linearised labour supply

$$\sigma \hat{c}_t + \psi \hat{n}_t = \hat{\omega}_t$$

- Log-linearised labour demand

$$\hat{a}_t + (\alpha - 1)\hat{n}_t = \hat{\omega}_t$$

Log-Linearisation

- Log-linearised Euler equation

$$\hat{c}_t = \mathbb{E}_t[\hat{c}_{t+1}] + \frac{1}{\sigma}(\hat{q}_t + \mathbb{E}_t[\hat{\pi}_{t+1}])$$

Recall that $q_t = 1/i_t \Rightarrow \hat{q}_t = -\hat{i}_t$.

$$\Rightarrow \hat{c}_t = \mathbb{E}_t[\hat{c}_{t+1}] + \frac{1}{\sigma}(-\hat{i}_t + \mathbb{E}_t[\hat{\pi}_{t+1}])$$

Recall the Fisher equation

$$i_t = r_t \mathbb{E}_t[\pi_{t+1}]$$

Implies that

$$\hat{i}_t = \hat{r}_t + \mathbb{E}_t[\hat{\pi}_{t+1}]$$

$$\Rightarrow \hat{c}_t = \mathbb{E}_t[\hat{c}_{t+1}] - \frac{1}{\sigma}\hat{r}_t$$

Log-Linearisation

- Log-linearised money demand (exercise, see appendix for derivation)

$$\hat{\mu}_{t+1} = \frac{\sigma}{\nu}\hat{c}_t - \left\{\frac{1}{\nu(\bar{i}-1)}\right\}\hat{i}_t$$

does it seem right? Increasing in $\hat{c}_t$ and decreasing in i_t?

- Log-linearised budget constraint

$$\bar{c}\hat{c}_t + \bar{\mu}\hat{\mu}_{t+1} = \bar{c}(\hat{\omega}_t + \hat{n}_t) + \bar{\mu}(\hat{\mu}_t - \hat{\pi}_t)$$

- Log-linearised money supply

$$\hat{\mu}_{t+1} = \hat{\mu}_t - \hat{\pi}_t + \zeta_{t+1}$$

Punch-Line: Short-Run Dynamics

- Deviations in nominal variables and real variables (excluding balances) are all separate. Why?

Roadmap

Steady State Nominal Variables

- Recall that $\bar{\pi} = 1$.
- Implies steady state in nominal prices $\bar{p}$.
- As well as steady state nominal balances $\bar{m}$.

Log-Linear Price Level

- Given that real and nominal variables are independent, set real variables to steady state.

- From money demand equation, see that

$$\begin{aligned}\hat{m}_{t+1} - \hat{p}_t &= -\frac{1}{\nu(\bar{i}-1)}\hat{i}_t \\ &= -\frac{1}{\nu(\bar{i}-1)}[\mathbb{E}_t[\hat{p}_{t+1}] - \hat{p}_t] \\ \Rightarrow \hat{p}_t &= \frac{1}{\nu(\bar{i}-1)+1}\mathbb{E}_t[\hat{p}_{t+1}] + \frac{\nu(\bar{i}-1)}{\nu(\bar{i}-1)+1}\hat{m}_{t+1}\end{aligned}$$

where the second line follows from the Fisher equation with the real rate at steady state.

Log-Linear Price Level

- Iterating forwards, we get that

$$\hat{p}_t = \frac{\nu(\bar{i} - 1)}{\nu(\bar{i} - 1) + 1} \mathbb{E}_t \left\{ \sum_{k=0}^{\infty} \left(\frac{1}{\nu(\bar{i} - 1) + 1} \right)^k \hat{m}_{t+k+1} \right\}$$

- Notice that we can express the money supply as

$$\hat{m}_{t+1} = \hat{m}_t + \zeta_{t+1}$$

- Or at an arbitrary time $t + k + 1$

$$\begin{aligned} \hat{m}_{t+k+1} &= \hat{m}_{t+k} + \zeta_{t+k+1} \\ &= \hat{m}_{t+k-1} + \zeta_{t+k+1} + \zeta_{t+k} \\ &= \hat{m}_{t+1} + \sum_{j=2}^{k+1} \zeta_{t+j} \end{aligned}$$

where the last line comes through repeated substitutions.

Log-Linear Price Level: Punch Line

- Price level deviations can then be expressed as

$$\hat{p}_t = \frac{\nu(\bar{i}-1)}{\nu(\bar{i}-1)+1}\mathbb{E}_t\left\{\sum_{k=0}^{\infty}\left(\frac{1}{\nu(\bar{i}-1)+1}\right)^k\left[\hat{m}_{t+1}+\sum_{j=2}^{k+1}\zeta_{t+j}\right]\right\}$$

- Can you simplify this any further? Exercise.
- The price level today is an infinite sum of expected monetary shocks in the future.
- Can then back-out other nominal variables like inflation and nominal interest rate (exercise).

Roadmap

Social Planner's Problem

- Efficient allocation achieved through maximising welfare subject only to technology (just like last class).

- Social planner's problem is

$$\max_{\{c_t, n_t, \mu_{t+1}\}_{t=0}^{\infty}} \mathbb{E}_0 \sum_{t=0}^{\infty} \beta^t \left[\frac{c_t^{1-\sigma}}{1-\sigma} + \frac{(\mu_{t+1})^{1-\nu}}{1-\nu} - \frac{n_t^{1+\psi}}{1+\psi} \right]$$

subject to technology $c_t = a_t n_t^{1-\alpha}$.

Social Planner's Problem: Optimality

- Lagrangian

$$\mathcal{L} = \mathbb{E}_0 \sum_{t=0}^{\infty} \beta^t \left[\frac{c_t^{1-\sigma}}{1-\sigma} + \frac{(\mu_{t+1})^{1-\nu}}{1-\nu} - \frac{n_t^{1+\psi}}{1+\psi} \right] + \mathbb{E}_0 \sum_{t=0}^{\infty} \lambda_t [a_t n_t^{1-\alpha} - c_t]$$

- FOCs

$$\frac{\partial \mathcal{L}}{\partial c_t} = 0 \Rightarrow \beta^t c_t^{-\sigma} - \lambda_t = 0 \tag{9}$$

$$\frac{\partial \mathcal{L}}{\partial n_t} = 0 \Rightarrow -\beta^t n_t^{\psi} + \lambda_t (1-\alpha) a_t n_t^{-\alpha} = 0 \tag{10}$$

$$\frac{\partial \mathcal{L}}{\partial \mu_{t+1}} = 0 \Rightarrow \beta^t \mu_{t+1}^{-\nu} = 0 \tag{11}$$

Friedman Rule

- How do we achieve (11) in the decentralised setting?
- Recall that $\mu_{t+1}^{-\nu} c_t^{\sigma} = 1 - q_t$ in decentralised setting.
- To get $\mu_{t+1}^{-\nu} = 0$, we need for $1 - q_t = 0$.
- That is: if $q_t = 1$, (no discount on bonds).
- Thus $i_t = 1$ (no net interest).
- Called the Friedman rule.

Friedman Rule

- Social marginal cost of creating extra money is zero.
- Private marginal cost is zero when there is a zero nominal interest rate.
- When $i_t = 1$, see from the Fisher equation that

$$\mathbb{E}_t[\pi_{t+1}] = \frac{1}{r_t}$$

- Those who hold money suffer no losses in its value due to inflation.

Roadmap

Takeaways

- MIU model is a first attempt at creating a market for money.
- Cheap as it just changes preferences.
- Clean as nominal variables are independent of real variables.
- Price level deviations simply a future forecast of monetary shocks.
- Optimal policy is the Friedman rule.

Log-Linearisation of Complicated Function Tricks

- Say we want to liniearise $y_t = f(x_t)$. See that

$$
\begin{aligned}
f(x_t) &\approx f(\bar{x}) + f'(\bar{x})(x_t - \bar{x}) \\
\Rightarrow f(x_t) - f(\bar{x}) &= f'(\bar{x})(x_t - \bar{x}) \\
\Rightarrow \frac{f(x_t) - f(\bar{x})}{f(\bar{x})} &= \frac{f'(\bar{x})}{f(\bar{x})}(x_t - \bar{x}) \\
\Rightarrow \hat{y}_t &= \frac{f'(\bar{x})}{f(\bar{x})}\bar{x}\hat{x}_t
\end{aligned}
$$

Log-Linearisation of Complicated Function Tricks

- Now see that $\mu_{t+1} = u_t c_t^{\sigma/\nu}$ where $u_t = \left(\frac{i_t}{i_t - 1}\right)^{\frac{1}{\nu}}$ for money demand.

- See that $\hat{u}_t = -\frac{1}{\nu(\bar{i}-1)}\hat{i}_t$.

- For general function $z_t = f(x_t, y_t)$, see that (exercise, show it)

$$\bar{z}\hat{z}_t = f_x(\bar{x}, \bar{y})\bar{x}\hat{x} + f_y(\bar{x}, \bar{y})\bar{y}\hat{y}$$

  Applying to the money demand gives

$$\hat{\mu}_{t+1} = \frac{1}{\bar{\mu}}\bar{c}^{\frac{\sigma}{\nu}}\left(\frac{\bar{i}}{\bar{i}-1}\right)^{\frac{1}{\nu}}\left[\frac{\sigma}{\nu}\hat{c}_t - \left\{\frac{1}{\nu(\bar{i}-1)}\right\}\hat{i}_t\right]$$

  Then use the steady state relationship to get the final expression.