Lecture 2: Real Business Cycle Model

Adam Hal Spencer

The University of Nottingham

Advanced Monetary Economics 2018

Roadmap

1. Introduction
2. RBC Environment
3. RBC Equilibrium
4. Big Picture
5. Steady State Analysis
6. Log-Linearised System
7. Solution
8. Conclusion

DSGE Models

- DSGE: Dynamic Stochastic General Equilibrium.
- Optimising representative agents and rational expectations.
- Controversial.
- Policy implications? Predictors of crises?
- Lots of work since the crisis on incorporating financial frictions.
- Last few years: relax the representative agent assumption; how do changes at the cross section affect aggregates?
- A quirky yet scathing review can be found in Quiggin (2010), "Zombie Economics: How Dead Ideas Still Walk Among Us".

Background

- RBC models are the original DSGEs.
- Combine microfoundations, dynamics and stochastic shocks to provide a theory of business cycle fluctuations.
- Consistent with the basic neoclassical growth model in the long-run.
- Exogeneous shocks (good assumption?) drive short-run fluctuations.
- Brock, W., & Mirman, L. (1972): "Optimal Economic Growth and Uncertainty: The Discounted Case", *Journal of Economic Theory*, 4(3), 479 513.
- Kydland, F & Prescott, E. (1982) "Time to Build and Aggregate Fluctuations", *Econometrica*, 50: 1345-1370.

Preview of the Punch-Line

- Business cycles are a natural part of life.
- Business cycles are efficient: can eventuate even without any market failures.
- Decentralised market equilibrium achieves the efficient allocation of resources.
- Business cycles are endogenous fluctuations, which are induced by shocks coming from external forces.
- Role of government should not be on smoothing business cycles: focus instead on structural reforms.

Roadmap

1. Introduction
2. RBC Environment
3. RBC Equilibrium
4. Big Picture
5. Steady State Analysis
6. Log-Linearised System
7. Solution
8. Conclusion

Fundamentals

- Representative agents: firms and households.
- Infinite horizon and discrete time $t \in \{0, 1, 2, 3, ...\}$.
- Perfectly competitive markets.
- General equilibrium.
- Real model: no role for money.
- Prices are denoted in terms of real variables (e.g. goods or labour).

Households Setup

- Supply labour to firms and own the capital stock, (rented out to firms).
- Objective is to maximise the expected present value of their lifetime utility subject to period-by-period budget constraints.
- Discounting over time: constant discount factor $0 < \beta < 1$, (money tomorrow is worth less than money today due to opportunity cost).
- Time separable utility.
- Household owns the firm and receives its profits as income d_t.

Households' Problem

- Problem:

$$\max_{\{c_t, n_t, i_t\}_{t=0}^{\infty}} \mathbb{E}_0 \sum_{t=0}^{\infty} \beta^t \left[\frac{c_t^{1-\sigma}}{1-\sigma} - \frac{n_t^{1+\psi}}{1+\psi} \right]$$

subject to their budget constraints and law of motion for capital

$$\begin{aligned} c_t + i_t &\leq w_t n_t + r_t k_t + d_t \\ k_{t+1} &= i_t + (1-\delta) k_t \\ k_{t+1} &\geq 0 \ \forall t \\ k_0 &\text{ given} \end{aligned}$$

- How does this differ from the infinite horizon optimisation problem from last class?

Firms' Problem

- Static problem since they rent factor inputs:

$$\max_{\{k_t, n_t\}} d_t = y_t - w_t n_t - r_t k_t$$

where $y_t = a_t k_t^{\alpha} n_t^{1-\alpha}$ and

$$\log(a_t) = \rho \log(a_{t-1}) + \epsilon_t, \;\; \epsilon_t \sim N(0,1)$$

where $0 < \rho < 1$.

- Zero profits $d_t = 0$.

Roadmap

1. Introduction
2. RBC Environment
3. RBC Equilibrium
4. Big Picture
5. Steady State Analysis
6. Log-Linearised System
7. Solution
8. Conclusion

Household Optimality: Lagrangian

- Lagrangian (substitute out investment for capital law of motion)

$$\mathcal{L} = \mathbb{E}_0 \sum_{t=0}^{\infty} \beta^t \left[\frac{c_t^{1-\sigma}}{1-\sigma} - \frac{n_t^{1+\psi}}{1+\psi} \right] +$$

$$\mathbb{E}_0 \sum_{t=0}^{\infty} \lambda_t \left[w_t n_t + (1-\delta+r_t)k_t + d_t - c_t - k_{t+1} \right]$$

Household Optimality: First Order Conditions

- Notice that $\mathbb{E}_t[x_t] = x_t$.
- FOCs:

$$\frac{\partial \mathcal{L}}{\partial c_t} = 0 \Rightarrow \beta^t c_t^{-\sigma} - \lambda_t = 0 \tag{1}$$

$$\frac{\partial \mathcal{L}}{\partial n_t} = 0 \Rightarrow -\beta^t n_t^{\psi} + \lambda_t w_t = 0 \tag{2}$$

$$\frac{\partial \mathcal{L}}{\partial k_{t+1}} = 0 \Rightarrow -\beta^t \lambda_t + \beta^{t+1}\mathbb{E}_t[\lambda_{t+1}(1 - \delta + r_{t+1})] = 0 \tag{3}$$

Firm Optimality: First Order Conditions

- FOCs:

$$\frac{\partial d_t}{\partial k_t} = 0 \Rightarrow \alpha a_t k_t^{\alpha-1} n_t^{1-\alpha} - r_t = 0 \tag{4}$$

$$\frac{\partial d_t}{\partial n_t} = 0 \Rightarrow (1-\alpha) a_t k_t^{\alpha} n_t^{-\alpha} - w_t = 0 \tag{5}$$

Equilibrium Definition

- The competitive equilibrium of the RBC model is defined as a sequence of prices $\{w_t, r_t\}_{t=0}^{\infty}$ and allocations $\{c_t, k_{t+1}, n_t\}$ with the state vector $\{k_t, a_t\}$ taken as given by the agents in the model. Optimality conditions (1) – (5) above hold, the no ponzi scheme condition holds and all markets clear.

Canonical Representation

- Consolidate the household's FOCs to get labour supply and consumption Euler equation. Resource constraint from household's budget constraint.
- (1) and (2) give labour supply

$$c_t^{\sigma} n_t^{\psi} = w_t$$

- (3) and (1) give the consumption Euler equation

$$1 = \beta \mathbb{E}_t \left[\left(\frac{c_{t+1}}{c_t} \right)^{-\sigma} (1 - \delta + r_{t+1}) \right]$$

- Household budget constraint, $d_t = 0$ and (4) – (5) give the resource constraint

$$c_t + i_t = w_t n_t + r_t k_t = y_t$$

- Look familiar?....

Social Planner's Problem

- Social planner's problem:

$$\max_{\{c_t, n_t, i_t\}_{t=0}^{\infty}} \mathbb{E}_0 \sum_{t=0}^{\infty} \beta^t \left[\frac{c_t^{1-\sigma}}{1-\sigma} - \frac{n_t^{1+\psi}}{1+\psi} \right]$$

subject to their budget constraints and law of motion for capital

$$\begin{aligned} c_t + i_t &= y_t \\ k_{t+1} &= i_t + (1-\delta) k_t \\ k_{t+1} &\geq 0 \ \forall t \\ k_0 & \text{ given} \end{aligned}$$

- The solution to this program is Pareto optimal.
- Exercise: show that this program yields the same solution as the RBC market economy above.

Roadmap

1. Introduction
2. RBC Environment
3. RBC Equilibrium
4. Big Picture
5. Steady State Analysis
6. Log-Linearised System
7. Solution
8. Conclusion

What's Going on in this Model?

- The solution is Pareto optimal, yet random shocks are still present.
- The productivity process a_t drives everything in this model!
- It's exogenous: philosophical implication?
- Shocks to productivity drive endogenous responses in other variables.
- We'll study local (small) deviations from a steady state as the solution.

Roadmap

1. Introduction
2. RBC Environment
3. RBC Equilibrium
4. Big Picture
5. Steady State Analysis
6. Log-Linearised System
7. Solution
8. Conclusion

What's a Steady State?

- The steady state of a model is defined as a situation in which variables are unchanging over time.
- In this model, this means $a_t = a_{t-1} = 1$ (as $\rho < 1$).
- As a consequence, $c_t = c_{t-1} = \bar{c}$, $k_t = k_{t-1} = \bar{k}$ etc for endogenous variables.
- The steady state is what prevails when we shut-down all the randomness in the model.

What Does the Steady State Look Like?

- Steady state labour supply

$$\bar{c}^{\sigma}\,\bar{n}^{\psi} = \bar{w} \tag{6}$$

- Steady state Euler equation

$$1 = \beta(1 - \delta + \bar{r}) \tag{7}$$

- Steady state resource constraint

$$\bar{c} + \bar{i} = \bar{y} \tag{8}$$

What Does the Steady State Look Like?

- From (5) and (6), the steady state factor prices are

$$\bar{r} = \alpha \bar{k}^{\alpha-1} \bar{n}^{1-\alpha} \tag{9}$$

$$\bar{w} = (1-\alpha) \bar{k}^{\alpha} \bar{n}^{-\alpha} \tag{10}$$

- From the capital law of motion, steady state investment is

$$\bar{i} = \delta \bar{k} \tag{11}$$

- From the production function, steady state output it

$$\bar{y} = \bar{k}^{\alpha} \bar{n}^{1-\alpha} \tag{12}$$

- From the technological process

$$\bar{a} = 1 \tag{13}$$

What Does the Steady State Look Like?

- Equations (6) – (13) define the steady state.
- Eight equations in eight unknowns $\{\bar{c}, \bar{n}, \bar{w}, \bar{r}, \bar{i}, \bar{y}, \bar{k}, \bar{a}\}$.
- Now we approximate small deviations about the steady state when shocks are present using log-linearisation.

Roadmap

1. Introduction
2. RBC Environment
3. RBC Equilibrium
4. Big Picture
5. Steady State Analysis
6. Log-Linearised System
7. Solution
8. Conclusion

Log-Linearisation

- Linearised labour supply

$$
\begin{aligned}
[\bar{c}e^{\hat{c}_t}]^{\sigma}[\bar{n}e^{\hat{n}_t}]^{\psi} &= \bar{w}e^{\hat{w}_t} \\
\Rightarrow \bar{c}^{\sigma}\bar{n}^{\psi}[e^{\sigma\hat{c}_t+\psi\hat{n}_t}] &= \bar{w}e^{\hat{w}_t} \\
\Rightarrow e^{\sigma\hat{c}_t+\psi\hat{n}_t} &= \bar{w}e^{\hat{w}_t} \\
\Rightarrow 1+\sigma\hat{c}_t+\psi\hat{n}_t &\approx 1+\hat{w}_t \\
\Rightarrow \sigma\hat{c}_t+\psi\hat{n}_t &\approx \hat{w}_t
\end{aligned}
\tag{14}
$$

where the first line comes from the definition of $\hat{x}_t$ [see lecture 1] and the penultimate line comes from a Taylor expansion of first order of line 3.

Log-Linearisation

- Linearised Euler equation

$$\hat{c}_t = \mathbb{E}_t[\hat{c}_{t+1}] + \frac{\bar{r}}{\sigma(1-\delta)}\mathbb{E}_t[\hat{r}_{t+1}] \tag{15}$$

- Linearised resource constraint

$$\hat{y}_t = \frac{\bar{c}}{\bar{y}}\hat{c}_t + \frac{\bar{i}}{\bar{y}}\hat{i}_t \tag{16}$$

- Linearised factor prices

$$\hat{r}_t = \hat{a}_t + (\alpha - 1)\hat{k}_t + (1-\alpha)\hat{n}_t \tag{17}$$

$$\hat{w}_t = \hat{a}_t + (\alpha)\hat{k}_t + (-\alpha)\hat{n}_t \tag{18}$$

Log-Linearisation

- Linearised capital law of motion

$$\hat{k}_{t+1} = (1-\delta)\hat{k}_t + \delta\hat{i}_t \tag{19}$$

- Linearised production function

$$\hat{y}_t = \hat{a}_t + \alpha\hat{k}_t + (1-\alpha)\hat{n}_t \tag{20}$$

- Linearised technology process

$$\hat{a}_t = \rho\hat{a}_{t-1} + \epsilon_t \tag{21}$$

Log-Linearisation

- Exercise: derive (15) – (21) yourself.
- Again we have eight equations in eight unknowns $\{\hat{c}_t, \hat{n}_t, \hat{w}_t, \hat{r}_t, \hat{i}_t, \hat{y}_t, \hat{k}_t, \hat{a}_t\}$.

Roadmap

1. Introduction
2. RBC Environment
3. RBC Equilibrium
4. Big Picture
5. Steady State Analysis
6. Log-Linearised System
7. Solution
8. Conclusion

How Do We Solve this System?

- Two approaches we can follow:
  (1) Try to solve the model analytically.
  (2) Solve it numerically.
- Analytical solution can only be found in really simple models.
- In general, we'll need to find a numerical solution.
- I'll teach you about numerical solutions when we get to the new Keynesian model, for now let's try the analytical approach.

State Variables

- The set of state variables of a model completely summarises the state of the dynamic system.
- If we know the state vector and the shocks realised for the current period, we can solve for the full system.
- The state variables in this model are $\hat{k}_t$ and $\hat{a}_t$. Why?
- Every other endogenous variable will be a function of these two.

Reduce the System

- Through substitutions, we can reduce the eight equation system down to four (exercise)

$$\hat{n}_t = \frac{1}{1+\psi}[\hat{a}_t + \alpha \hat{k}_t - \sigma \hat{c}_t] \tag{22}$$

$$\hat{c}_t = \mathbb{E}_t[\hat{c}_{t+1}] + \frac{\bar{r}}{\sigma(1-\delta)}\mathbb{E}_t[\hat{a}_{t+1} + (\alpha-1)\hat{k}_{t+1} + (1-\alpha)\hat{n}_{t+1}] \tag{23}$$

$$(1-\alpha)\hat{n}_t = \frac{\bar{c}}{\bar{y}}\hat{c}_t + \frac{\bar{i}}{\bar{y}}\frac{1}{\delta}[\hat{k}_{t+1} - (1-\delta)\hat{k}_t] - \hat{a}_t - \alpha \hat{k}_t \tag{24}$$

$$\hat{a}_t = \rho \hat{a}_{t-1} + \epsilon_t \tag{25}$$

which are in four variables $\{\hat{n}_t, \hat{k}_{t+1}, \hat{a}_t, \hat{c}_t\}$.

Method of Undetermined Coefficients

- We can guess policy functions of the form

$$\hat{n}_t = \eta_{n,a}\hat{a}_t + \eta_{n,k}\hat{k}_t \tag{26}$$

$$\hat{k}_{t+1} = \eta_{k',a}\hat{a}_t + \eta_{k',k}\hat{k}_t \tag{27}$$

$$\hat{c}_t = \eta_{c,a}\hat{a}_t + \eta_{c,k}\hat{k}_t \tag{28}$$

  where each of these controls are functions of the state variables.

- Recall: the state variables determine the entire system.

Method of Undetermined Coefficients

- Start with (23), substitute-in (26) for $\hat{n}_{t+1}$

$$\hat{c}_t = \mathbb{E}_t[\hat{c}_{t+1}] + \frac{\bar{r}}{\sigma(1-\delta)}\mathbb{E}_t[\hat{a}_{t+1} + (\alpha-1)\hat{k}_{t+1} + (1-\alpha)\{\eta_{n,a}\hat{a}_{t+1} + \eta_{n,k}\hat{k}_{t+1}\}]$$

$$= \mathbb{E}_t[\hat{c}_{t+1}] + \frac{\bar{r}}{\sigma(1-\delta)}\mathbb{E}_t[(1+\{1-\alpha\}\eta_{n,a})\hat{a}_{t+1} + (1-\alpha)\{\eta_{n,k}-1\}\hat{k}_{t+1}]$$

$$= \mathbb{E}_t[\eta_{c,a}\hat{a}_{t+1} + \eta_{c,k}\hat{k}_{t+1}] + \frac{\bar{r}}{\sigma(1-\delta)}\mathbb{E}_t[(1+\{1-\alpha\}\eta_{n,a})\hat{a}_{t+1} + (1-\alpha)\{\eta_{n,k}-1\}\hat{k}_{t+1}]$$

Method of Undetermined Coefficients

$$= \left[\eta_{c,a} + \frac{\bar{r}}{\sigma(1-\delta)}[(1+\{1-\alpha\}\eta_{n,a})\right]\mathbb{E}_t[\hat{a}_{t+1}] + \left[\eta_{c,k} + \frac{\bar{r}}{\sigma(1-\delta)}\{1-\alpha\}(\eta_{n,k}-1)\right]\hat{k}_{t+1} \tag{29}$$

$$\Rightarrow \hat{c}_t = \left[\eta_{c,a} + \frac{\bar{r}}{\sigma(1-\delta)}[(1+\{1-\alpha\}\eta_{n,a})\right]\rho\hat{a}_t + \left[\eta_{c,k} + \frac{\bar{r}}{\sigma(1-\delta)}\{1-\alpha\}(\eta_{n,k}-1)\right]\hat{k}_{t+1}$$

Method of Undetermined Coefficients

- Re-arrange (24) to get

$$\hat{k}_{t+1} = \frac{\bar{y}}{\bar{i}}\delta \left\{ (1-\alpha)\hat{n}_t - \frac{\bar{c}}{\bar{y}}\hat{c}_t + \hat{a}_t + \left[\alpha + \frac{\bar{i}}{\bar{y}}\frac{1}{\delta}(1-\delta) \right] \hat{k}_t \right\}$$

substitute-in (26) and (28) to yield

$$\begin{aligned} \hat{k}_{t+1} = & \frac{\bar{y}}{\bar{i}}\delta \left\{ 1 + (1-\alpha)\eta_{n,a} - \frac{\bar{c}}{\bar{y}}\eta_{c,a} \right\} \hat{a}_t + \\ & + \frac{\bar{y}}{\bar{i}}\delta \left\{ \alpha + \frac{\bar{i}}{\bar{y}}\frac{1}{\delta}(1-\delta) + (1-\alpha)\eta_{n,k} - \frac{\bar{c}}{\bar{y}}\eta_{c,k} \right\} \hat{k}_t \end{aligned} \quad (30)$$

Method of Undetermined Coefficients

- Substitute (28) into (22) for

$$\hat{n}_t = \frac{1}{1+\psi}\left[(1-\sigma\eta_{c,a})\hat{a}_t + (\alpha-\sigma\eta_{c,k})\hat{k}_t\right] \tag{31}$$

Method of Undetermined Coefficients

- Equate (31) with (26) to yield

$$\eta_{n,a}\hat{a}_t + \eta_{n,k}\hat{k}_t = \frac{1}{1+\psi}(1-\sigma\eta_{c,a})\hat{a}_t + \frac{1}{1+\psi}(\alpha-\sigma\eta_{c,k})\hat{k}_t$$

- Now equate the coefficients to get

$$\eta_{n,a} = \frac{1}{1+\psi}(1-\sigma\eta_{c,a})$$
$$\eta_{n,k} = \frac{1}{1+\psi}(\alpha-\sigma\eta_{c,k}).$$

- Repeating this for the other equations will yield 6 linear equations in 6 unknowns, $\{\eta_{n,a}, \eta_{n,k}, \eta_{c,a}, \eta_{c,k}, \eta_{k',a}, \eta_{k',k}\}$, (exercise).

Roadmap

1. Introduction
2. RBC Environment
3. RBC Equilibrium
4. Big Picture
5. Steady State Analysis
6. Log-Linearised System
7. Solution
8. Conclusion

Back to the Big Picture

- This system of variables respond endogenously to productivity shocks.
- E.g. say we start in steady state at $t = 0$ and then $\hat{a}_1 = \epsilon_1$ then no further shocks (called an impulse response).
- We can trace-out the time paths for the endogenous variables:

$$\hat{n}_1 = \eta_{n,a}\epsilon_1$$
$$\hat{k}_2 = \eta_{k',a}\epsilon_1$$
$$\hat{c}_1 = \eta_{c,a}\epsilon_1$$

$$\hat{n}_2 = \eta_{n,a}[\rho\epsilon_1] + \eta_{n,k}[\eta_{k',a}\epsilon_1]$$
$$\hat{k}_3 = \eta_{k,a}[\rho\epsilon_1] + \eta_{k',k}[\eta_{k',a}\epsilon_1]$$
$$\hat{c}_2 = \eta_{c,a}[\rho\epsilon_1] + \eta_{c,k}[\eta_{k',a}\epsilon_1]$$

......

Back to the Big Picture

- Under certain stability conditions for the parameters (to be discussed later), we'll eventually converge back to steady state.