Lecture 1: Introduction and Mathematical Preliminaries

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Advanced Monetary Economics 2018

Roadmap

1. Introduction
2. Static Optimisation
3. Dynamic Discrete Time Optimisation
4. Stochastic Models
5. Log-Linearisation
6. Summary

Instructor

- Adam Spencer
  - No need for formalities: call me either Adam or Spencer.
- Assistant Professor of Economics (started here this September).
- Ph.D. Economics and Finance, M.S. Economics.
  - University of Wisconsin-Madison (USA).
- M.Econ. (Hons), B.Comm. (Hons) Economics.
  - The University of Melbourne (Australia).

Course Overview

- Objectives:
  (1) Equip you with the tools required to do research in monetary economics.
  (2) Apply the rigour of these tools to studying policy and other topical issues.
  (3) Continue to develop your economic intuition.
  (4) Bridge the gap between earlier intuition you've developed in earlier courses and modelling techniques.

Summary

- The material covered in this course will be tough!
- You'll get exposure to lots of new things: may seem intimidating.
- Look through all the math to see the intuition of models and solutions.
- This is not a math course!

Note

- These mathematical methods are just recipes that I want you to know how to use.
- Again, this is not a math course: these are just tools for doing economics.

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Constrained Optimisation

- "Economics is the study of how society manages its scarce resources" (Mankiw, 2007, *Principles of Economics*).
- Constrained optimisation!

Static Program

- A static optimisation program will have the following general form

$$\max_{\vec{x}} \; f(\vec{x}, p) \text{ s.t. } g(\vec{x}, p) = \gamma$$

- This will have the following Lagrangian

$$\mathcal{L} = f(\vec{x}, p) + \lambda[\gamma - g(\vec{x}, p)]$$

  where $\lambda \geq 0$ is called the Lagrange multiplier.

- Interior solution found by taking $\frac{\partial \mathcal{L}}{\partial x_i}$ for all $x_i \in \vec{x}$ and $\frac{\partial \mathcal{L}}{\partial \lambda}$ and equating the derivatives with zero (first order conditions).

- We'll focus just on interior solutions, (corner solutions require the use of Kuhn-Tucker conditions).

Static Optimisation Example

- Solve the following consumption-leisure tradeoff problem:

$$\max_{c,n} \ \frac{c^{1-\sigma}}{1-\sigma} - \chi n$$

subject to $c = wn$ where w is taken as given.

Static Optimisation Example Solution (1)

- Lagrangian given by

$$\mathcal{L} = \frac{c^{1-\sigma}}{1-\sigma} - \chi n + \lambda[wn - c]$$

- First order conditions (FOCs) given by

$$\frac{\partial \mathcal{L}}{\partial c} = 0 \Rightarrow c^{-\sigma} - \lambda = 0 \quad (1)$$

$$\frac{\partial \mathcal{L}}{\partial n} = 0 \Rightarrow -\chi + \lambda w = 0 \quad (2)$$

$$\frac{\partial \mathcal{L}}{\partial \lambda} = 0 \Rightarrow wn - c = 0 \quad (3)$$

Static Optimisation Example Solution (2)

- Equations (1) and (2) imply

$$c^{-\sigma} = \frac{\chi}{w} \Rightarrow c = \left(\frac{\chi}{w}\right)^{-\frac{1}{\sigma}} \tag{4}$$

- Plug (4) into (3) to get the solution for n as

$$n = \left(\frac{\chi}{w}\right)^{-\frac{1}{\sigma}} / w$$

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Discrete Time Deterministic Program

- Consider a problem of the form

$$\max_{\vec{x}_t} \sum_{t=0}^{\infty} f(\vec{x}_t, p, t) \text{ s.t. } g(\vec{x}_t, p, t) = \gamma_t \ \forall t \geq 0$$

- Has the Lagrangian

$$\mathcal{L} = \sum_{t=0}^{\infty} f(\vec{x}_t, p, t) + \sum_{t=0}^{\infty} \lambda_t[\gamma_t - g(\vec{x}_t, p, t)]$$

where $\lambda_t \geq 0$ are the Lagrange multipliers.

Discrete Time Optimisation Example

- Solve the following program

$$\max_{\{c_t\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t \left[\frac{c_t^{1-\sigma}}{1-\sigma} - \chi n_t \right]$$

for $\beta \in [0, 1]$ subject to the constraint

$$p_t c_t + q_t b_t = b_{t-1} + w_t n_t$$

where b_t are discount bonds, $(q_t < 1)$ and the price sequences $\{w_t, p_t, q_t\}_{t=0}^{\infty}$ are taken as given.

- Notice that the dynamics have an effect through savings, b_t.

Discrete Time Optimisation Example Solution (1)

- Lagrangian given by

$$\mathcal{L} = \sum_{t=0}^{\infty} \beta^t \left[\frac{c_t^{1-\sigma}}{1-\sigma} - \chi n_t \right] + \sum_{t=0}^{\infty} \lambda_t [b_{t-1} + w_t n_t - p_t c_t - q_t b_t]$$

which comes with first order conditions

$$\frac{\partial \mathcal{L}}{\partial c_t} = 0 \Rightarrow \beta^t c_t^{-\sigma} - \lambda_t p_t = 0 \tag{5}$$

$$\frac{\partial \mathcal{L}}{\partial n_t} = 0 \Rightarrow -\beta^t \chi + \lambda_t w_t = 0 \tag{6}$$

$$\frac{\partial \mathcal{L}}{\partial b_t} = 0 \Rightarrow -q_t \lambda_t + \lambda_{t+1} = 0 \tag{7}$$

$$\frac{\partial \mathcal{L}}{\partial \lambda_t} = 0 \Rightarrow p_t c_t + q_t b_t = b_{t-1} + w_t n_t, \tag{8}$$

Discrete Time Optimisation Example Solution (2)

- Recall that the price sequences are all taken as given (exogenous).
- Using (5) and (6) yields

$$c_t^{-\sigma} = \frac{\chi}{w_t/p_t} \Rightarrow c_t = \left(\frac{w_t}{p_t}\frac{1}{\chi}\right)^{\sigma} \tag{9}$$

- FOC (5) tells us that

$$\lambda_t = \frac{\beta^t c_t^{-\sigma}}{p_t} \tag{10}$$

Discrete Time Optimisation Example Solution (3)

- Combining (7) and (10) yields

$$\beta \left(\frac{c_{t+1}}{c_t} \right)^{-\sigma} \frac{p_t}{p_{t+1}} = q_t \tag{11}$$

  which is referred to as a consumption Euler equation.

- Equations (11), (9) and (8) together summarise the solution to the program.

Discrete Time Optimisation Example Solution (4)

- Solution to the program is given by a sequence $\{c_t, n_t, b_t\}_{t=0}^{\infty}$ that satisfies

$$\beta \left(\frac{c_{t+1}}{c_t} \right)^{-\sigma} \frac{p_t}{p_{t+1}} = q_t$$

$$c_t = \left(\frac{w_t}{p_t} \frac{1}{\chi} \right)^{\sigma}$$

$$p_t c_t + q_t b_t = b_{t-1} + w_t n_t$$

together with initial condition b_{-1} and "no ponzi game" restriction

$$\lim_{t \to \infty} \left[\prod_{j=0}^{t} q_j \right] b_t = 0$$

which says that the NPV of the "terminal" asset holdings are zero.

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Shocks

- The examples we've looked at so far were all deterministic.
- What happens when we add random shocks to the model?
- Control variables will be a function of realised state of the world.

Randomness and States of Nature

- In this course, we'll assume that there is an information set that evolves over time denoted by $\mathcal{I}_t$.
- In the future, there is some set of possible outcomes $\omega_i \in \Omega$.
- All the agents in the model know the set Ω for the future, they just don't know what ω_i will come up.
- Take expectations over the states and form state-contingent plans for control variables.
- $\mathbb{E}_t[x]$ is shorthand for $\mathbb{E}[x|\mathcal{I}_t]$

Two Period Stochastic Model Example

- Consider an optimal savings problem for a consumer over two periods $t \in \{0, 1\}$.
- The consumer receives endowment of income y_t in period t where $y_t = \bar{y} + \epsilon_t$ where $\mathbb{E}[\epsilon_t] = 0$.
- Consumer maximises NPV of expected lifetime utility where period utility function is $\frac{c_t^{1-\sigma}}{1-\sigma}$.
- Assume that price of consumption in each period is unity and bond price is fixed at q_0.
- Variables will all be functions of the state realised at decision time $\omega_t \in \Omega$.

Two Period Stochastic Model Example

- The consumer is faced with the problem:

$$\max_{c_0(\omega_0),c_1(\omega_1),b_0(\omega_0)} \mathbb{E}_0\left[\frac{c_0((\omega_0))^{1-\sigma}}{1-\sigma}+\beta\frac{c_1((\omega_1))^{1-\sigma}}{1-\sigma}\right]$$

subject to

$$c_0(\omega_0)+q_0 b_0(\omega_0)=y_0(\omega_0)$$
$$c_1(\omega_1)=b_0(\omega_0)+y_1(\omega_1)$$

Two Period Stochastic Model Example Solution

- Objective given by,

$$\mathcal{L} = \mathbb{E}_0 \left[\frac{(y_0(\omega_0) - q_0 b_0(\omega_0))^{1-\sigma}}{1-\sigma} + \beta \frac{(b_0(\omega_0) + y_1(\omega_1))^{1-\sigma}}{1-\sigma} \right]$$

  which is a function of only one control b_0 from substituting out c_0 and c_1.

- Optimality condition given by

$$\frac{d\mathcal{L}}{db_0} = 0 \Rightarrow q_0 c_0(\omega_0)^{-\sigma} = \beta \mathbb{E}_0[c_1^{-\sigma}(\omega_1)]$$

  which is a stochastic consumption Euler equation.

- See that the optimal decision depends on the state realised at $t = 0$ and what's expected at $t = 1$.

Two Period Stochastic Model Example Solution

- Can we solve $q_0 c_0(\omega_0)^{-\sigma} = \beta \mathbb{E}_0[c_1^{-\sigma}(\omega_1)]$ for $b_0(\omega_0)$ in closed form?
- No! Either use numerical methods or local approximations.
- As is canonical in monetary economics, we'll use lots of local approximations through the log-linearisation technique.
- Note: from now on, I'll drop the state scripts to ease notation (i.e. y_0 rather than $y_0(\omega_0)$).

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Steady State

- We reach a steady state when nothing is changing.
- In the previous example, this is given by $\epsilon_t = 0$ for each period.
- That is: $y_0 = y_1 = \bar{y}$.
- Other variables will be unchanging as well $c_0 = c_1 = \bar{c}$.

Steady State Example

- In the steady state of the two-period model, see

$$q_0\bar{c}^{-\sigma} = \beta\bar{c}^{-\sigma} \Rightarrow q_0 = \beta \tag{12}$$
$$\bar{c} = \bar{y} - q_0 b_0 \tag{13}$$
$$\bar{c} = \bar{y} + \bar{b_0}. \tag{14}$$

- For (12) – (14) to all hold, we need for no savings (i.e. $b_0 = 0$) between periods.

- Follows that $\bar{c} = \bar{y}$: consumption each period just equals the deterministic endowment.

Log-Linearisation

- Approximates non-linear solutions around the steady state.
- Define the log deviation of a variable (x_t) from its steady state as

$$\hat{x}_t = \log\left(\frac{x_t}{\bar{x}}\right)$$

Log-Linearisation

- We can interpret $\hat{x}_t$ as a percentage deviation of the variable from its steady state as:

$$\begin{aligned}
\hat{x}_t &= \log\left(\frac{x_t}{\bar{x}}\right) \\
&= \log\left(1 + \frac{x_t - \bar{x}}{\bar{x}}\right) \\
&= \log(1) + \frac{1}{\bar{x}}(x_t - \bar{x}) + \text{higher order terms} \\
&\approx \log(1) + \frac{1}{\bar{x}}(x_t - \bar{x}) \\
&= \frac{x_t - \bar{x}}{\bar{x}}
\end{aligned}$$

where the penultimate line follows if the deviation from $\bar{x}$ is small, (first order Taylor expansion is valid).

Log-Linearisation Example

- Linearise the consumption Euler equation from the two period model:

$$q_0 c_0^{-\sigma} = \beta \mathbb{E}_0[c_1^{-\sigma}]$$

Log-Linearisation Example Solution

- From our definition of the deviations, see that

$$\hat{c}_0 = \log\left(\frac{c_0}{\bar{c}}\right) \Rightarrow c_0 = \bar{c}e^{\hat{c}_0}.$$

- We can plug this into the Euler equation to get

$$\begin{aligned} q_0(\bar{c}e^{\hat{c}_0})^{-\sigma} &= \beta\mathbb{E}_0[(\bar{c}e^{\hat{c}_0})^{-\sigma}] \\ \Rightarrow q_0\bar{c}^{-\sigma}e^{-\sigma\hat{c}_0} &= \beta\bar{c}^{-\sigma}\mathbb{E}_0[e^{-\sigma\hat{c}_1}] \\ \Rightarrow e^{-\sigma\hat{c}_0} &= \mathbb{E}_0[e^{-\sigma\hat{c}_1}] \\ \Rightarrow (1-\sigma\hat{c}_0) &= \mathbb{E}_0[(1-\sigma\hat{c}_1)] \\ \Rightarrow \hat{c}_0 &= \mathbb{E}_0[\hat{c}_1] \end{aligned}$$

where the third line comes from steady state equation (12) and the fourth line comes from a first order Taylor expansion of $\exp(1+x)$.

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Topics Covered

- These mathematical techniques are just tools.
- If you understand how to implement all these methods today, you'll be good for the math in this first half of the course.