Lecture 1: Mathematical Methods

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Advanced Financial Economics 2020

Roadmap

Instructor

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- Assistant Professor of Economics (started here September 2018).
- Ph.D. Economics and Finance, M.S. Economics.
  - University of Wisconsin-Madison (USA).
- M.Econ. (Hons), B.Comm. (Hons) Economics.
  - The University of Melbourne (Australia).

Course Overview

- The course is split into two parts.
  - **(1)** Corporate finance.
  - **(2)** Asset pricing.

(1) Corporate finance

- Say a firm wants to take a new project.
- Corporate finance asks the question of how they best finance the new project?
- In this part of the course, we seek to answer two questions:
  - (A) What does theory predict the optimal financing mix should be?
  - (B) Which factors matter most quantitatively (in the data)?

(2) Asset pricing

- Here we consider
  - (A) How do we characterise asset prices using microfoundations?
  - (B) When, in the real world, can we say that there is an asset price bubble?

Tools

- This course covers a lot of ground and thus requires a lot of different tools.
- We'll make theoretical predictions, test them with data and leverage both simultaneously using structural models.

Summary

- The material covered in this course will be tough!
- It will be MATHEMATICAL IN NATURE.
- You'll get exposure to lots of new things: may seem intimidating.
- Look through all the math to see the intuition of models and solutions.
- This is not a math course!

Roadmap

NPV Analysis

- When evaluating a project, we use NPV analysis.
- Takes the cash flows associated with the project, discounts them and adds them together.
- Why is discounting necessary?
- Opportunity cost of time: instead of investing in this project, I could take the funds and stick them into a riskless bank account and get interest earnings.
- Interest earnings are the opportunity cost.

Discount rate and discount factor

- What's the appropriate opportunity cost to use?
- Depends on our investors.
- What is their opportunity cost? What is their discount factor?
- Assume that time in the world is discrete $t \in \{0, 1, 2, 3, ...\}$.
- If their net opportunity cost is $r > 0$ per period ($1 + r$ gross return), then how to we discount cash flows for them?

Discount rate and discount factor

- I.e. what is £1 received at $t+1$ worth in terms of time t money?
- Invest £1 in this bank account at $t \Rightarrow$ get back £(1+r) at $t+1$. We seek x in the following

$$1 \text{ at } t \Rightarrow (1+r) \text{ at } t+1$$
$$x \text{ at } t \Rightarrow 1 \text{ at } t+1$$

  which yields $x = \frac{1}{1+r}$.

- I.e. £1 received at $t+1$ is worth $\frac{1}{1+r}$ at time t.
- The sooner we get money, the better! We can do more with it!
- Object r is referred to as the discount rate.
- An object defined as $\beta = \frac{1}{1+r}$ is referred to as the discount factor.

Example

- E.g. consider a project that Firm A is contemplating taking. The project has an upfront cost of $c_0 > 0$ and then generates $c > 0$ in positive cash flows in perpetuity from $t = 1$ onwards. The investors in the firm can invest in a riskless bank account that offers $r > 0$ of net interest per period. What is the NPV of this project?

$$
\begin{aligned}
NPV &= -c_0 + \frac{1}{1+r}c + \left(\frac{1}{1+r}\right)^2 c + \left(\frac{1}{1+r}\right)^3 c + \ldots \\
&= -c_0 + \sum_{t=0}^{\infty} \left(\frac{1}{1+r}\right)^t c \\
&= -c_0 + \sum_{t=0}^{\infty} \beta^t c \\
&= -c_0 + \frac{c}{1-\beta}
\end{aligned}
$$

where the penultimate line follows from the definition of the discount factor and the last line comes from geometric series.

Example

- E.g. When will the example on the previous slide constitute a good project from the perspective of the investors? When the NPV is positive!

$$-c_0 + \frac{c}{1-\beta} \geq 0$$
$$\Rightarrow c_0 \leq \frac{c}{1-\beta}$$
$$\Rightarrow \beta \geq \frac{c}{c_0} - 1$$

what does this mean?

- Says that the project is good only if the investors are sufficiently patient. Make sense?

- Notice that this evaluates the project relative to our next best alternative.

NPV versus utility

- This symbol β hopefully looks somewhat familiar in this context.
- Recall lifetime utility is often given as

$$\sum_{t=0}^{\infty} \beta^t u(c_t)$$

where $u(c_t)$ is referred to as the period utility function and c_t is consumption for period t.

- A risk averse household has $u(c_t)$ concave in consumption, (i.e. $u' > 0$ and $u'' < 0$).
- A risk neutral household has $u(c_t)$ as being linear in consumption.

NPV versus utility

- Say there is no upfront cost for a project.
- And that it pays out a sequence $\{c_t\}_{t=0}^{\infty}$

$$NPV = \sum_{t=0}^{\infty} \beta^t c_t$$

which says that NPV is the utility associated with consuming the cash flows for a risk-neutral investor.

- In corporate finance, we'll typically assume risk neutral investors.
- Not the case in asset pricing though (more on this later in the semester).

Roadmap

Constrained Optimisation

- "Economics is the study of how society manages its scarce resources" (Mankiw, 2007, *Principles of Economics*).
- Constrained optimisation!

Discrete Time Deterministic Program

- Consider a problem of the form

$$\max_{\vec{x}_t} \sum_{t=0}^{\infty} f(\vec{x}_t, p, t) \text{ s.t. } g(\vec{x}_t, p, t) = \gamma_t \ \forall t \geq 0$$

- Has the Lagrangian

$$\mathcal{L} = \sum_{t=0}^{\infty} f(\vec{x}_t, p, t) + \sum_{t=0}^{\infty} \lambda_t [\gamma_t - g(\vec{x}_t, p, t)]$$

where $\lambda_t \geq 0$ are the Lagrange multipliers.

Dynamic Optimisation Example

- Solve the following program for a risk averse household

$$\max_{\{c_t, b_{t+1}\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t \frac{c_t^{1-\sigma}}{1-\sigma}$$

for $\beta \in [0, 1]$ subject to the constraint

$$c_t + q_t b_{t+1} = b_t + y_t$$

and with b_0 given. See that b_t are discount bonds, y_t is their income endowment, $q_t < 1$ is the bond price and the price sequence $\{q_t\}_{t=0}^{\infty}$ is taken as given.

- Notice that the dynamics have an effect through savings, b_t.

- What are the control variables here?

Dynamic Optimisation Example Solution (1)

- Lagrangian given by

$$\mathcal{L} = \sum_{t=0}^{\infty} \beta^t \frac{c_t^{1-\sigma}}{1-\sigma} + \sum_{t=0}^{\infty} \lambda_t [b_t + y_t - c_t - q_t b_{t+1}]$$

which comes with first order conditions

$$\frac{\partial \mathcal{L}}{\partial c_t} = 0 \Rightarrow \beta^t c_t^{-\sigma} - \lambda_t = 0 \quad (1)$$

$$\frac{\partial \mathcal{L}}{\partial b_{t+1}} = 0 \Rightarrow -q_t \lambda_t + \lambda_{t+1} = 0 \quad (2)$$

$$\frac{\partial \mathcal{L}}{\partial \lambda_t} = 0 \Rightarrow c_t + q_t b_{t+1} = b_t + y_t, \quad (3)$$

Discrete Time Optimisation Example Solution (2)

- Combining (1) and (2) yields

$$\beta \left(\frac{c_{t+1}}{c_t} \right)^{-\sigma} = q_t \tag{4}$$

which is referred to as a consumption Euler equation.

- Equations (3) and (4) together summarise the solution to the program.

Roadmap

Shocks

- The examples we've looked at so far were all deterministic.
- What happens when we add random shocks to the model?
- Control variables will be a function of realised state of the world.

Randomness and States of Nature

- In this course, we'll assume that there is an information set that evolves over time denoted by $\mathcal{I}_t$.
- In the future, there is some set of possible outcomes $\omega_i \in \Omega$.
- All the agents in the model know the set Ω for the future, they just don't know what ω_i will come up.
- Take expectations over the states and form state-contingent plans for control variables.
- $\mathbb{E}_t[x]$ is shorthand for $\mathbb{E}[x|\mathcal{I}_t]$

Two Period Stochastic Model Example

- Consider an optimal savings problem for a consumer over two periods $t \in \{0, 1\}$.
- The consumer receives endowment of income y_t in period t where $y_t = \bar{y} + \epsilon_t$ where $\mathbb{E}[\epsilon_t] = 0$.
- Consumer maximises NPV of expected lifetime utility where period utility function is $\frac{c_t^{1-\sigma}}{1-\sigma}$.
- Assume that price of consumption in each period is unity and bond price is fixed at q_0.
- Variables will all be functions of the state realised at decision time $\omega_t \in \Omega$.

Two Period Stochastic Model Example

- The consumer is faced with the problem:

$$\max_{c_0(\omega_0),c_1(\omega_1),b_0(\omega_0)} \mathbb{E}_0\left[\frac{c_0((\omega_0))^{1-\sigma}}{1-\sigma}+\beta\frac{c_1((\omega_1))^{1-\sigma}}{1-\sigma}\right]$$

subject to

$$\begin{aligned} c_0(\omega_0)+q_0 b_0(\omega_0) &= y_0(\omega_0) \\ c_1(\omega_1) &= b_0(\omega_0)+y_1(\omega_1) \end{aligned}$$

Two Period Stochastic Model Example Solution

- Objective given by,

$$\mathcal{L} = \mathbb{E}_0 \left[\frac{(y_0(\omega_0) - q_0 b_0(\omega_0))^{1-\sigma}}{1-\sigma} + \beta \frac{(b_0(\omega_0) + y_1(\omega_1))^{1-\sigma}}{1-\sigma} \right]$$

  which is a function of only one control b_0 from substituting out c_0 and c_1.

- Optimality condition given by

$$\frac{d\mathcal{L}}{db_0(\omega_0)} = 0 \Rightarrow q_0 c_0(\omega_0)^{-\sigma} = \beta \mathbb{E}_0[c_1^{-\sigma}(\omega_1)]$$

  which is a stochastic consumption Euler equation.

- See that the optimal decision depends on the state realised at $t = 0$ and what's expected at $t = 1$.

Roadmap

Topics Covered

- These mathematical techniques are just tools.
- If you understand how to implement all these methods today, you'll be good for the basic techniques needed for this module.