Lecture 11: Theory of Asset Pricing II
Portfolio Choice

Adam Hal Spencer

The University of Nottingham

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Roadmap

1. Introduction
2. Quadratic Utility Portfolio Choice
3. Minimum Variance Frontier
4. Conclusion

Motivation

- So far in the asset pricing part of the course, we've thought about the consumption-savings decision.
- Where savings will take place through a risky asset.
- How should investors optimally allocate their optimal savings across multiple assets?
- Portfolio theory.

Motivation

- This is a very old area of research that we'll study today.
- It's an intuitive concept, but gets quite algebra-intensive.
- Easy to get lost in math rather than thinking about economics.
- For this reason, we'll only spend one lecture on it.
- We'll proceed in two steps: firstly thinking about a general portfolio choice problem, then into a more specific case.

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Quadratic utility

- The canonical model of Markowitz (1952) assumes quadratic utility in wealth.
- If we make this assumption, then we get an intuitive trade-off for the portfolio choice problem: the investor trades-off expected returns against variance.

Quadratic utility

- We'll think about a static model here, (just one time period).
- Abstract from thinking about the consumption-savings tradeoff.
- The investor has some amount of wealth W that they wish to invest.
- Assume they consume all their wealth at the end of the time period.
- Meaning that their utility function is over the total amount of wealth they have (after the returns to their investments are realised).

Quadratic utility

- The utility function is of the form

$$U(W) = aW - \frac{b}{2}W^2$$

where W denotes wealth. In expected utility terms, see that

$$\begin{aligned}\mathbb{E}U(W) &= a\mathbb{E}[W] - \frac{b}{2}\mathbb{E}[W^2] \\ &= a\mathbb{E}[W] - \frac{b}{2}\{\mathbb{E}[W]\}^2 - \frac{b}{2}\text{Var}(W).\end{aligned}$$

- Do we need any more assumptions for this utility function to make sense?
- Place assumptions on W such that expected utility is always increasing in expected wealth.
- Doesn't really make sense to think that welfare is decreasing in expected wealth.

Quadratic utility

- This utility function is nice.
- We can think of maximising this expected utility as maximising $\mathbb{E}(W)$ for a given $\text{Var}(W)$ or minimising variance for a given expected wealth.

Quadratic utility

- There's a neat implication of this utility function.
- We can scrap thinking about utility functions all together when making our portfolio choice.
- If we minimise variance to meet a certain expected return threshold, then we're maximising the investor's utility (assuming that it's quadratic like here).

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Two risky assets

- This is where the algebra becomes nightmarish.
- We'll take a simple approach: assume two risky assets: denote their returns r_A and r_B.
- Denote their expected returns μ_A and μ_B and variances σ_A^2 and σ_B^2.
- Assume (to get a trade-off), that $\mu_A > \mu_B$ but $\sigma_A^2 > \sigma_B^2$.
- I.e. there is not dominating asset.

Two risky assets

- Let's choose portfolio holdings (α_A, α_B) in each of the two assets to minimise portfolio return variance subject to a required return.
- We'll also impose here that $\alpha_A + \alpha_B = 1$.
- Assume also that the two returns are independent of each other.
- The portfolio return is $r_p = \alpha_A r_A + \alpha_B r_B$. Means that
  - $\mathbb{E}[r_p] = \alpha_A \mu_A + \alpha_B \mu_B$.
  - $\text{Var}(r_p) = \alpha_A^2 \sigma_A^2 + \alpha_B^2 \sigma_B^2$.

Two risky assets

- Investor's problem is then

$$\min_{\{\alpha_A,\alpha_B\}} \; \frac{1}{2}(\alpha_A^2\sigma_A^2 + \alpha_B^2\sigma_B^2)$$

subject to

$$\begin{aligned} \alpha_A\mu_A + \alpha_B\mu_B &= \bar{\mu} \\ \alpha_A + \alpha_B &= 1 \end{aligned}$$

where $\bar{\mu}$ is the investor's required expected return.

- Why do I put the half in the objective?

- This problem says: we're minimising the variance of our portfolio subject to a certain expected return requirement.

Two risky assets

- Investor's Lagrangian is

$$\mathcal{L} = \frac{1}{2}(\alpha_A^2\sigma_A^2 + \alpha_B^2\sigma_B^2) + \lambda[\alpha_A\mu_A + \alpha_B\mu_B - \bar{\mu}] + \gamma[1 - \alpha_A - \alpha_B]$$

with FOCs

$$\frac{\partial \mathcal{L}}{\partial \alpha_i} = 0$$
$$\Rightarrow \sigma_i^2\alpha_i + \lambda\mu_i - \gamma = 0$$
$$\Rightarrow \alpha_i = \frac{\lambda\mu_i + \gamma}{\sigma_i^2}$$

which holds for both $i \in \{A, B\}$.

Two risky assets

- Where to from here?
- We want expressions for the Lagrange multipliers λ and γ (endogenous) as functions of the parameters (exogenous).
- The two constraints will bind: use these!

Two risky assets

- See that

$$\alpha_A = \frac{\lambda \mu_A + \gamma}{\sigma_A^2}$$
$$\alpha_B = \frac{\lambda \mu_B + \gamma}{\sigma_B^2}$$

Two risky assets

- Recall these two weights sum to one

$$
\begin{aligned}
\frac{\lambda\mu_A + \gamma}{\sigma_A^2} + \frac{\lambda\mu_B + \gamma}{\sigma_B^2} &= 1 \\
\Rightarrow (\lambda\mu_A + \gamma)\sigma_B^2 + (\lambda\mu_B + \gamma)\sigma_A^2 &= \sigma_A^2\sigma_B^2 \\
\Rightarrow (\mu_A\sigma_B^2 + \mu_B\sigma_A^2)\lambda + (\sigma_A^2 + \sigma_B^2)\gamma &= \sigma_A^2\sigma_B^2 \\
\Rightarrow \lambda &= \frac{\sigma_A^2\sigma_B^2 - \gamma(\sigma_A^2 + \sigma_B^2)}{\mu_A\sigma_B^2 + \mu_B\sigma_A^2} \\
&= \frac{1 - \gamma\left[\frac{1}{\sigma_A^2} + \frac{1}{\sigma_B^2}\right]}{\frac{\mu_A}{\sigma_A^2} + \frac{\mu_B}{\sigma_B^2}} \qquad (1)
\end{aligned}
$$

Two risky assets

- Then we have the other constraint for the required expected return

$$\frac{\lambda\mu_A + \gamma}{\sigma_A^2}\mu_A + \frac{\lambda\mu_B + \gamma}{\sigma_B^2}\mu_B = \bar{\mu}$$
$$\Rightarrow \mu_A(\lambda\mu_A + \gamma)\sigma_B^2 + \mu_B(\lambda\mu_B + \gamma)\sigma_A^2 = \bar{\mu}\sigma_A^2\sigma_B^2$$
$$\Rightarrow \lambda[\mu_A^2\sigma_B^2 + \mu_B^2\sigma_A^2] + [\mu_A\sigma_B^2 + \mu_B\sigma_A^2]\gamma = \bar{\mu}\sigma_A^2\sigma_B^2$$
$$\Rightarrow \gamma = \frac{\bar{\mu}\sigma_A^2\sigma_B^2 - \lambda[\mu_A^2\sigma_B^2 + \mu_B^2\sigma_A^2]}{[\mu_A\sigma_B^2 + \mu_B\sigma_A^2]} \quad (2)$$

Two risky assets

- Finally we can combine (1) and (2) to get

$$\lambda = \frac{1 - \left\{ \frac{\bar{\mu}\sigma_A^2\sigma_B^2 - \lambda[\mu_A^2\sigma_B^2 + \mu_B^2\sigma_A^2]}{[\mu_A\sigma_B^2 + \mu_B\sigma_A^2]} \right\} \left[\frac{1}{\sigma_A^2} + \frac{1}{\sigma_B^2} \right]}{\frac{\mu_A}{\sigma_A^2} + \frac{\mu_B}{\sigma_B^2}}$$

$$\Rightarrow \left\{ \frac{\mu_A}{\sigma_A^2} + \frac{\mu_B}{\sigma_B^2} \right\} \lambda = 1 - \left\{ \frac{\bar{\mu}\sigma_A^2\sigma_B^2 - \lambda[\mu_A^2\sigma_B^2 + \mu_B^2\sigma_A^2]}{[\mu_A\sigma_B^2 + \mu_B\sigma_A^2]} \right\} \left[\frac{1}{\sigma_A^2} + \frac{1}{\sigma_B^2} \right]$$

$$\Rightarrow \left(\frac{\mu_A}{\sigma_A^2} + \frac{\mu_B}{\sigma_B^2} - \left[\frac{1}{\sigma_A^2} + \frac{1}{\sigma_B^2} \right] \frac{\mu_A^2\sigma_B^2 + \mu_B^2\sigma_A^2}{\mu_A\sigma_B^2 + \mu_B\sigma_A^2} \right) \lambda$$

$$= 1 - \left[\frac{1}{\sigma_A^2} + \frac{1}{\sigma_B^2} \right] \frac{\bar{\mu}\sigma_A^2\sigma_B^2}{\mu_A\sigma_B^2 + \mu_B\sigma_A^2}$$

Two risky assets

- Hence

$$\lambda = \frac{1 - \left[\frac{1}{\sigma_A^2} + \frac{1}{\sigma_B^2}\right] \frac{\bar{\mu}\sigma_A^2\sigma_B^2}{\mu_A\sigma_B^2 + \mu_B\sigma_A^2}}{\left(\frac{\mu_A}{\sigma_A^2} + \frac{\mu_B}{\sigma_B^2} - \left[\frac{1}{\sigma_A^2} + \frac{1}{\sigma_B^2}\right] \frac{\mu_A^2\sigma_B^2 + \mu_B^2\sigma_A^2}{\mu_A\sigma_B^2 + \mu_B\sigma_A^2}\right)}$$

where γ comes from plugging this into (2).

Two risky assets

- What are these objects?
- The Lagrange multipliers can be written in terms of the variances, expected returns and required expected return on the portfolio!

Two risky assets

- What is this object α_i that we've found?
- Tell me three things and I can tell you the optimal weight α_i:
  - The required portfolio return: $\bar{\mu}$.
  - Asset A details: (μ_A, σ_A^2).
  - Asset B details: (μ_B, σ_B^2).
- The solution is referred to as the minimum variance frontier.

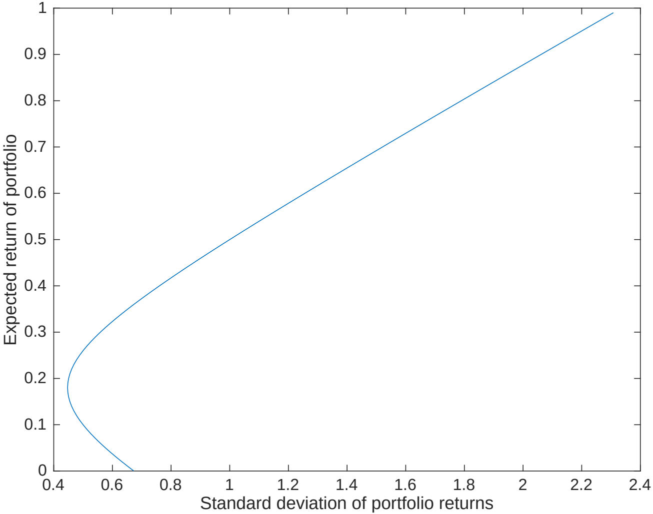
Two risky assets

- From there, we have α_A and α_B.
- We know the expected return on the portfolio is given by $\bar{\mu}$.
- The portfolio return variance is then $\alpha_A^2\sigma_A^2 + \alpha_B^2\sigma_B^2$.
- For two given risky assets, what does the optimal solution look like in expected return-variance space?
- I.e. if we took the expressions for $\gamma, \lambda, \alpha_A, \alpha_B$ and found the corresponding variance.
- It's a mess analytically, we can draw it numerically though.

Two risky assets

- Set $(\mu_A, \sigma_A^2) = (0.5, 1.0)$ and $(\mu_B, \sigma_B^2) = (0.1, 0.25)$.
- How does the portfolio variance change with the portfolio required/expected return?
- The numbers on this slide and the next are not examinable, but the shape of the MVF in $\bar{\mu}$ and σ space is examinable.

Two risky assets



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Summary

- In this lecture we've talked about how one should allocate their wealth amongst different assets.
- Under the assumption of quadratic utility, we get the minimum variance frontier (MVF).
- The MVF embodies this idea that we like returns but dislike risk.