Lecture 10: Theory of Asset Pricing I
Consumption-Based Asset Pricing

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Roadmap

What is Asset Pricing?

- So far we've talked about firms and their investment/financing behaviour.
- We can think of a firm as an asset.
- It offers cash flows (to its owner) each period as a sequence $\{d_t\}_{t=0}^{\infty}$.
- The question is: what is this sequence of cash flows worth today and at any given point in the future?

What is Asset Pricing?

- In our corporate finance setting, the value of the firm was given by

$$\mathbb{E}_0 \sum_{t=0}^{\infty} \beta^t d_t.$$

- This was the value of the dividend stream to the owners.
- What does β account for? Time-value of money/opportunity cost.
- What assumptions went into this? Risk neutrality of the owners!

What is Asset Pricing?

- Notice the expectation term though: $\mathbb{E}_0$.
- This assumption of risk neutrality is not without loss of generality when there is randomness to the firm's cash flows.
- How should we deal with risk aversion?
- What is the value of dividend streams like this more generally?

Roadmap

Terminology

- An asset offers a stream of dividends of $\{d_t\}_{t=0}^{\infty}$.
- We'll denote the initial price of this asset by p_0.
- The return on this asset at time $t = 1$ is denoted by r_1.
- The three are then related by

$$r_1 = \frac{p_1 + d_1}{p_0}$$

- More generally

$$r_{t+1} = \frac{p_{t+1} + d_{t+1}}{p_t}$$

is called the gross return on the asset at time $t + 1$.

Roadmap

Consumption asset pricing

- Consider a two-period endowment economy model for a consumer.
- Time periods are $t \in \{0, 1\}$.
- The household is endowed with e_0 and e_1 of consumption goods in the first and second periods respectively.
- They have period utility function given by $u(c_t)$ where c_t is their consumption at time t.
- Discount time $t = 1$ utility with discount factor β.

Consumption asset pricing

- Assume that they can save through holdings of a risky asset.
- This risky asset has price p_0 at time $t = 0$ and offers a stochastic dividend of d_1 at time $t = 1$ (and nothing thereafter).
- They choose how much of the risky asset to take into the future (denoted a_1).

Consumer's problem

- Solves the problem

$$\max_{c_0, c_1, a_1} \quad u(c_0) + \beta \mathbb{E}_0[u(c_1)]$$

where

$$c_0 = e_0 - p_0 a_1$$
$$c_1 = e_1 + d_1 a_1$$

Consumer's problem

- Substitute-in the constraints to get objective

$$\mathcal{L} = u(e_0 - p_0 a_1) + \beta \mathbb{E}_0[u(e_1 + d_1 a_1)]$$

which has FOC given by

$$\begin{aligned} \frac{\partial \mathcal{L}}{\partial a_1} &= 0 \\ \Rightarrow u'(c_0)p_0 + \beta \mathbb{E}_0[u'(c_1)d_1] &= 0 \\ \Rightarrow p_0 &= \beta \mathbb{E}_0 \left[\frac{u'(c_1)d_1}{u'(c_0)} \right] \end{aligned}$$

Consumer's solution

- Notice that the price and dividend are something that the consumer takes as given.
- Their consumption sequence satisfies

$$p_0 = \beta \mathbb{E}_0 \left[\frac{u'(c_1) d_1}{u'(c_0)} \right].$$

Consumer's solution

- More generally, what happens when we have an infinite-horizon model with the same setup.
- That is — imagine that the household gets some endowment sequence $\{e_t\}_{t=0}^{\infty}$ and seeks to choose asset holding positions $\{a_{t+1}\}_{t=0}^{\infty}$ to maximise their lifetime utility $\mathbb{E}_0 \sum_{t=0}^{\infty} \beta^t u(c_t)$.

Consumer's solution

- Consumer solves the problem

$$\max_{\{a_{t+1},c_t\}_{t=0}^{\infty}} \quad \mathbb{E}_0 \sum_{t=0}^{\infty} \beta^t u(c_t)$$

where

$$c_t + p_t a_{t+1} = e_t + (p_t + d_t) a_t$$

what's going on in this budget constraint?

- Household re-balances their asset holdings each period.

Consumer's solution

- Lagrangian

$$\mathcal{L} = \mathbb{E}_0 \sum_{t=0}^{\infty} \beta^t u(c_t) + \mathbb{E}_0 \sum_{t=0}^{\infty} \lambda_t [e_t + (p_t + d_t) a_t - c_t - p_t a_{t+1}]$$

with FOCs

$$\frac{\partial \mathcal{L}}{\partial c_t} = 0 \Rightarrow \beta^t u'(c_t) - \lambda_t = 0$$

$$\frac{\partial \mathcal{L}}{\partial a_{t+1}} = 0 \Rightarrow -\lambda_t p_t + \mathbb{E}_t[\lambda_{t+1}(p_{t+1} + a_{t+1})] = 0$$

Consumer's solution

- Where we can combine these two FOCs to get the Euler equation

$$p_t = \beta \mathbb{E}_t \left[\frac{u'(c_{t+1})}{u'(c_t)} \{ p_{t+1} + d_{t+1} \} \right]$$

- How does this compare with our corporate finance lectures?

Asset pricing v.s. corporate finance

- Recall that risk neutrality implies that $u'(c_t)$ is a constant (since the utility function is linear).

- The Euler equation then becomes

$$p_t = \beta \mathbb{E}_t \left[p_{t+1} + d_{t+1} \right] \tag{1}$$

iterating forwards by one period gives

$$p_{t+1} = \beta \mathbb{E}_{t+1} \left[p_{t+2} + d_{t+2} \right] \tag{2}$$

where I've just updated the time indices by one period.

Asset pricing v.s. corporate finance

- Substitute (2) into (1) to get

$$\begin{aligned} p_t &= \beta \mathbb{E}_t \left[\beta \mathbb{E}_{t+1} \left[p_{t+2} + d_{t+2} \right] + d_{t+1} \right] \\ &= \beta^2 \mathbb{E}_t [p_{t+2}] + \beta^2 \mathbb{E}_t [d_{t+2}] + \beta \mathbb{E}_t [d_{t+1}] \end{aligned}$$

  where the second line uses the law of iterated expectations (i.e. that $\mathbb{E}_t[\mathbb{E}_{t+1}[x_{t+2}]] = \mathbb{E}_t[x_{t+2}]$)

Asset pricing v.s. corporate finance

- Continuing in this way forever yields

$$p_t = \mathbb{E}_t \sum_{j=0}^{\infty} \beta^{t+j+1} d_{t+j+1} + \lim_{T \to \infty} \beta^T \mathbb{E}_t[p_T].$$

- Assume that the last term goes to zero, (otherwise the discounted price explodes). Gives

$$p_t = \mathbb{E}_t \sum_{j=0}^{\infty} \beta^{t+j+1} d_{t+j+1}.$$

Asset pricing v.s. corporate finance

- Or for time zero onwards

$$p_0 = \mathbb{E}_0 \sum_{t=0}^{\infty} \beta^{t+1} d_{t+1}$$

- If we add period zero's dividend to either side we get

$$p_0 + d_0 = \mathbb{E}_0 \sum_{t=0}^{\infty} \beta^{t} d_{t}$$

  which is the firm's objective!

- The left-side is the total value of the firm's equity at time $t = 0$ going onwards.

Asset pricing v.s. corporate finance

- In general, when we have a risk averse owner of the firm, the firm's objective would look like

$$\mathbb{E}_0 \sum_{t=0}^{\infty} \beta^t \frac{u'(c_t)}{u'(c_0)} d_t$$

- Why?

- This accounts for not just the time-value of money, but also accounts for an uncertain path of consumption going forward.

- $\beta^t \frac{u'(c_t(\omega))}{u'(c_0)}$ tells me how much the risk averse owner likes consumption in state ω at time t relative to consumption at time zero.

- Need to weight the dividend flows accordingly.

Stochastic discount factor

- Recall the consumption-based asset pricing equation

$$p_t = \mathbb{E}_t \left[\beta \frac{u'(c_{t+1})}{u'(c_t)} \{ p_{t+1} + d_{t+1} \} \right]$$

- The object $\beta \frac{u'(c_{t+1})}{u'(c_t)}$ is referred to as the period-ahead stochastic discount factor.

Roadmap

Returns

- From our definition of returns, we can re-write this consumption-based asset pricing equation as

$$p_t = \beta \mathbb{E}_t \left[\frac{u'(c_{t+1})}{u'(c_t)} \{p_{t+1} + d_{t+1}\} \right]$$
$$\Rightarrow 1 = \beta \mathbb{E}_t \left[\frac{u'(c_{t+1})}{u'(c_t)} \frac{p_{t+1} + d_{t+1}}{p_t} \right]$$
$$\Rightarrow 1 = \beta \mathbb{E}_t \left[\frac{u'(c_{t+1})}{u'(c_t)} r_{t+1} \right]$$

for some return r_{t+1} on an arbitrary asset.

Riskless returns

- It will also often be convenient to think about the stochastic discount factor to the rate of return on a riskless asset.
- Since the asset is riskless, it will offer a certain return, that we'll denote r^f.
- See then that

$$1 = \beta \mathbb{E}_t \left[\frac{u'(c_{t+1})}{u'(c_t)} r^f \right]$$
$$\Rightarrow 1 = r^f \beta \mathbb{E}_t \left[\frac{u'(c_{t+1})}{u'(c_t)} \right]$$
$$\Rightarrow r^f = \frac{1}{\beta \mathbb{E}_t \left[\frac{u'(c_{t+1})}{u'(c_t)} \right]}$$

where the r^f comes outside the expectation since it's riskless.

Riskless returns

- The riskless return equals the reciprocal of the expected stochastic discount factor.
- The stochastic discount factor varies based on the realised state in the future!

Roadmap

Summary

- Micro-founded model of asset pricing based-on the consumption-savings decision of a household.