Lecture 10: Solving DSGE Models Part I Analytical Solution Methods

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Roadmap

Recap

- Last lecture: we derived the NK Phillips curve.
- Where are we at now with the new Keynesian model?
- All we've really talked about so far is the supply-side of the dynamic model.
- What about the demand-side and government?

Roadmap

Households

- We basically just take the household problem from lecture 7, (the static imperfect competition model), then embed a savings problem.
- Problem:

$$\max_{\{C_t, N_t, B_{t+1}\}_{t=0}^{\infty}} \mathbb{E}_0 \sum_{t=0}^{\infty} \beta^t \left[\frac{C_t^{1-\sigma}}{1-\sigma} - \frac{N_t^{1+\psi}}{1+\psi} \right]$$

subject to their budget constraint

$$P_t C_t + Q_t B_{t+1} \leqslant W_t N_t + B_t + D_t$$

where $C_t = \left(\int_0^1 C_t(j)^{\frac{\epsilon-1}{\epsilon}} dj \right)^{\frac{\epsilon}{\epsilon-1}}$

Households

- FOCs:

$$\beta^t C_t^{-\sigma} - \lambda_t P_t = 0$$

$$-\beta^t N_t^{\psi} + \lambda_t W_t = 0$$

$$-\lambda_t Q_t + \mathbb{E}_t[\lambda_{t+1}] = 0$$

Log-Linearised Form

- Labour supply and Euler equation (see lecture 4 for derivation):

$$\sigma \hat{c}_t + \psi \hat{n}_t = \hat{w}_t - \hat{p}_t$$
$$\hat{c}_t = \mathbb{E}_t[\hat{c}_{t+1}] - \frac{1}{\sigma}(\hat{i}_t - \mathbb{E}_t[\hat{\pi}_{t+1}]).$$

- Recall that we used the labour supply equation in deriving the new Keynesian Phillips curve.
- The Euler equation will form the foundation for the dynamic IS curve.

Log-Linearised Form

- Substitute the market clearing condition ($Y_t = C_t$) into the Euler equation to obtain

$$\hat{y}_t = \mathbb{E}_t[\hat{y}_{t+1}] - \frac{1}{\sigma}(\hat{i}_t - \mathbb{E}_t[\hat{\pi}_{t+1}]) \tag{1}$$

- Recall from the last lecture that the output gap is defined as

$$\hat{y}_t^g = \hat{y}_t - \hat{y}_t^n$$

where $\hat{y}_t^n$ is the natural level of output (flexible price equilibrium).

- Re-write (1) to get the dynamic IS curve

$$\hat{y}_t^g = \mathbb{E}_t[\hat{y}_{t+1}^g] - \frac{1}{\sigma}(\hat{i}_t - \mathbb{E}_t[\hat{\pi}_{t+1}] - \hat{r}_t^n)$$

where $\hat{r}_t^n = \sigma\mathbb{E}_t[\hat{y}_{t+1}^n - \hat{y}_t^n]$.

Dynamic IS Curve

- Or expressed differently

$$\hat{y}_t^g = \mathbb{E}_t[\hat{y}_{t+1}^g] - \frac{1}{\sigma}(\hat{r}_t - \hat{r}_t^n)$$

where

$$\hat{r}_t = \hat{i}_t - \mathbb{E}_t[\hat{\pi}_{t+1}]$$
$$\hat{y}_t^n = \mathbb{E}_t[\hat{y}_{t+1}^n] - \frac{1}{\sigma}\hat{r}_t^n$$

- That is — the output gap is proportional to the deviation in the real interest rate from its natural counterpart.

Dynamic IS Curve

- Recall that the traditional Keynesian IS curve plotted output as a function of the interest rate.
- This dynamic analogue looks at the growth rate in output relative to real interest rate.
- Equation (1) is probably the closest version to the traditional IS curve.
- We instead choose to present it in the form of the output gap to relate to the Phillips curve.

Monetary Authority

- Finally, we need to say something about the monetary authority.
- Controls the nominal interest rate: has real impact given rigid prices.
- Taylor rule: a reduced-form way of capturing monetary policy behaviour.

$$\hat{i}_t = \phi_\pi \hat{\pi}_t + \phi_y \hat{y}_t^g + \nu_t$$

  where ν_t is a monetary policy shock.

- Responses to both inflation and output gap captures the "dual mandate".

Full System

- Three equations in three unknowns $(\hat{\pi}_t, \hat{i}_t, \hat{y}_t^g)$

$$\hat{\pi}_t = \kappa \hat{y}_t^g + \beta \mathbb{E}_t[\hat{\pi}_{t+1}]$$
$$\hat{y}_t^g = \mathbb{E}_t[\hat{y}_{t+1}^g] - \frac{1}{\sigma}(\hat{i}_t - \mathbb{E}_t[\hat{\pi}_{t+1}] - \hat{r}_t^n)$$
$$\hat{i}_t = \phi_\pi \hat{\pi}_t + \phi_y \hat{y}_t^g + \nu_t$$

- Where exogenous processes are given by

$$\hat{a}_{t+1} = \rho_a \hat{a}_t + \epsilon_{t+1}^a, \;\; \epsilon_{t+1}^a \sim N(0, \sigma_a^2)$$
$$\hat{v}_{t+1} = \rho_v \hat{v}_t + \epsilon_{t+1}^v, \;\; \epsilon_{t+1}^v \sim N(0, \sigma_v^2)$$

- Where to from here?

- How do we solve this?

Roadmap

Solution

- We want to take this three equation system and express it in terms of the shocks in the model?
- How many shocks are in this model?
- Which variables are expressed in terms of which shocks?
- For this model we can find an analytical solution in terms of the parameters of the model using a method of "guess and verify".
- This works in this model since everything is simple.
- Won't necessarily work for bigger and more complicated models, (need numerical methods for this: next lecture).

Guess and Verify

- Guess that the endogenous variables are linearly impacted by shocks.
- Flexible price equilibrium variables are affected by $\hat{a}_t$ or $(\hat{a}_t, \hat{v}_t)$?
- What about the remaining variables? Why?

Guess and Verify

- Conjecture that

$$\hat{y}_t^n = \gamma_{ya}^n \hat{a}_t$$
$$\Rightarrow \hat{r}_t^n = \sigma \gamma_{ya}^n \mathbb{E}_t[\hat{a}_{t+1} - \hat{a}_t]$$

as well as

$$\hat{\pi}_t = \gamma_{\pi a} \hat{a}_t + \gamma_{\pi v} \hat{v}_t$$
$$\hat{y}_t^g = \gamma_{ya}^g \hat{a}_t + \gamma_{yv}^g \hat{v}_t.$$

Guess and Verify

- Iterating forwards then means that

$$\begin{aligned}\mathbb{E}_t[\hat{\pi}_{t+1}] &= \mathbb{E}_t[\gamma_{\pi a}\hat{a}_{t+1} + \gamma_{\pi v}\hat{v}_{t+1}] \\ &= \gamma_{\pi a}\rho_a\hat{a}_t + \gamma_{\pi v}\rho_v\hat{v}_t\end{aligned}$$

$$\begin{aligned}\mathbb{E}_t[\hat{y}^g_{t+1}] &= \mathbb{E}_t[\gamma^g_{ya}\hat{a}_{t+1} + \gamma^g_{yv}\hat{v}_{t+1}] \\ &= \gamma^g_{ya}\rho_a\hat{a}_t + \gamma^g_{yv}\rho_v\hat{v}_t\end{aligned}$$

$$\hat{r}^n_t = \sigma\gamma^n_{ya}(\rho_a - 1)\hat{a}_t$$

what is the intuition behind $\hat{r}^n_t$'s coefficient?

Guess and Verify: NK Phillips Curve

- Substitute these guesses into the new Keynesian Phillips curve

$$
\begin{aligned}
&\hat{\pi}_t = \kappa \hat{y}_t^g + \beta \mathbb{E}_t[\hat{\pi}_{t+1}] \\
&\Rightarrow (\gamma_{\pi a}\hat{a}_t + \gamma_{\pi v}\hat{v}_t) = \kappa(\gamma_{ya}^g \hat{a}_t + \gamma_{yv}^g \hat{v}_t) + \beta(\gamma_{\pi a}\rho_a \hat{a}_t + \gamma_{\pi v}\rho_v \hat{v}_t) \\
&\Rightarrow \hat{a}_t \left\{\kappa \gamma_{ya}^g + \beta \gamma_{\pi a}\rho_a - \gamma_{\pi a}\right\} + \hat{v}_t \left\{\kappa \gamma_{yv}^g + \beta \gamma_{\pi v}\rho_v - \gamma_{\pi v}\right\} = 0
\end{aligned}
$$

  notice that the last line must hold for any pair of realisations $(\hat{a}_t, \hat{v}_t)$.

- Follows that

$$
\begin{aligned}
\kappa \gamma_{ya}^g + \beta \gamma_{\pi a}\rho_a - \gamma_{\pi a} &= 0 \\
\kappa \gamma_{yv}^g + \beta \gamma_{\pi v}\rho_v - \gamma_{\pi v} &= 0
\end{aligned}
$$

Guess and Verify: Dynamic IS Curve

- Substitute the monetary rule and $\hat{r}_t^n$ in

$$\hat{y}_t^g = \mathbb{E}_t[\hat{y}_{t+1}^g] - \frac{1}{\sigma}(\hat{i}_t - \mathbb{E}_t[\hat{\pi}_{t+1}] - \hat{r}_t^n)$$

$$\Rightarrow \hat{y}_t^g = \mathbb{E}_t[\hat{y}_{t+1}^g] - \frac{1}{\sigma}(\phi_\pi \hat{\pi}_t + \phi_y \hat{y}_t^g + \nu_t - \mathbb{E}_t[\hat{\pi}_{t+1}] + \sigma\gamma_{ya}^n(\rho_a - 1)\hat{a}_t)$$

$$\Rightarrow \left\{\gamma_{ya}^g \left(1 + \frac{1}{\sigma}\phi_y - \rho_a\right) + \gamma_{\pi a}\frac{1}{\sigma}\left(\phi_\pi - \rho_a\right) + \gamma_{ya}^n(\rho_a - 1)\right\}\hat{a}_t +$$

$$\left\{\gamma_{yv}^g \left(1 + \frac{1}{\sigma}\phi_y - \rho_v\right) + \gamma_{\pi v}\frac{1}{\sigma}\left(\phi_\pi - \rho_v\right) + 1\right\}\hat{v}_t$$

- Follows that

$$\gamma_{ya}^g \left(1 + \frac{1}{\sigma}\phi_y - \rho_a\right) + \gamma_{\pi a}\frac{1}{\sigma}\left(\phi_\pi - \rho_a\right) + \gamma_{ya}^n(\rho_a - 1) = 0$$

$$\gamma_{yv}^g \left(1 + \frac{1}{\sigma}\phi_y - \rho_v\right) + \gamma_{\pi v}\frac{1}{\sigma}\left(\phi_\pi - \rho_v\right) + 1 = 0$$

Guess and Verify: System of Equations

- Four equations in four unknowns $(\gamma_{\pi v}, \gamma_{\pi a}, \gamma_{yv}^{g}, \gamma_{ya}^{g})$

$$
\begin{aligned}
-(1-\beta\rho_a)\gamma_{\pi a} + \kappa\gamma_{ya}^{g} &= 0 \\
-(1-\beta\rho_v)\gamma_{\pi v} + \kappa\gamma_{yv}^{g} &= 0 \\
\gamma_{\pi a}\frac{1}{\sigma}(\phi_\pi - \rho_a) + \gamma_{ya}^{g}\left(1 + \frac{1}{\sigma}\phi_y - \rho_a\right) &= -\gamma_{ya}^{n}(\rho_a - 1) \\
\gamma_{\pi v}\frac{1}{\sigma}(\phi_\pi - \rho_v) + \gamma_{yv}^{g}\left(1 + \frac{1}{\sigma}\phi_y - \rho_v\right) &= -1
\end{aligned}
$$

Guess and Verify: Matrix Form

$$\begin{bmatrix} 0 & -(1-\beta\rho_a) & 0 & \kappa \\ -(1-\beta\rho_v) & 0 & \kappa & 0 \\ 0 & \frac{1}{\sigma}\left(\phi_\pi - \rho_a\right) & \left(1+\frac{1}{\sigma}\phi_y - \rho_a\right) & 0 \\ \frac{1}{\sigma}\left(\phi_\pi - \rho_v\right) & 0 & \left(1+\frac{1}{\sigma}\phi_y - \rho_v\right) & 0 \end{bmatrix} \begin{bmatrix} \gamma_{\pi v} \\ \gamma_{\pi a} \\ \gamma_{yv}^{g} \\ \gamma_{ya}^{g} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ -\gamma_{ya}^{n}(\rho_a - 1) \\ -1 \end{bmatrix}$$

- Can in principle invert the matrix of coefficients, multiply by the vector of constants and get the coefficients.

- Can then back-out all the other variables.

Impulse Responses

- The system of variables will respond endogenously to shocks.
- Can trace-out the time paths followed by variables.
- E.g. say a one-time shock to monetary policy and then no further shocks.

$$
\begin{aligned}
\hat{\pi}_0 &= \gamma_{\pi v}\hat{v}_0 \\
\hat{\pi}_1 &= \gamma_{\pi v}\hat{v}_1 \\
&= \gamma_{\pi v}[\rho_v\hat{v}_0] \\
\hat{\pi}_1 &= \gamma_{\pi v}\hat{v}_2 \\
&= \gamma_{\pi v}[\rho_v^2\hat{v}_0] \\
&\ldots
\end{aligned}
$$

the resulting sequence of variables traced-out is known as an impulse response.

Roadmap

Takeaways

- If the model is simple enough, we can solve it analytically.
- Method of undetermined coefficients.
- If it's not sufficiently simple, we need to use numerical methods — next lecture.